# Chapter 1
# Analytic Fourier Theory Review

## 1.1 A Little History and Purpose

The branch of optical science known today as "Fourier optics" had its genesis in the 1940s through the 1960s with the application of new telecommunications and circuit design analysis techniques in optical diffraction theory.[1] In 1968 this upstart discipline was given a permanent foothold with the publication of *Introduction to Fourier Optics*, by Joseph W. Goodman, a seminal textbook that explained and united the fundamental concepts, and which continues to add significantly to the application of Fourier optics in subsequent editions.[2] Fourier optics is now the cornerstone for the analysis of diffraction, coherence, and imaging, as well as specialized topics such as wavefront control, propagation through random media, and holography.

The study of Fourier optics today leads naturally toward the computer for at least two reasons: (1) diffraction integral expressions are difficult to solve analytically for all but a few of the simplest aperture functions, and (2) the fast Fourier transform (FFT) algorithm combined with the linear systems framework of Fourier optics provides an extremely efficient computational approach for solving wave optics problems.

Certainly, the computer can be applied directly in finding exceedingly accurate solutions to diffraction problems using numerical integration techniques.[3] However, this book is really about the FFT and how to apply it to a variety of Fourier optics problems. The computer coding steps mirror the analytic concepts and the FFT's speed makes it possible to perform thousands of optical propagation or imaging simulations in a reasonable amount of time. In fact, the methods explored in this book form the basis for wave (or physical) optics simulation tools that are widely used in industry. But, of course, there's no free lunch (…if there were, perhaps we could be eating *while* studying Fourier optics...). It turns out the FFT is an accomplice to various numerical artifacts. We do our best in this book to expose these issues and provide constraints to help minimize the damage.

This is also a tutorial text with step-by-step instructions, not only for coding Fourier optics problems, but also for MATLAB, our software application of choice. So, if you are new to MATLAB, don't worry! Chapter 3 starts at the beginning ("Open MATLAB") and leads you through the basics of working with the FFT. By the end of the book you will be programming diffraction problems involving partially coherent light—at least that's the goal! Exercises at the end of the chapters give you room to tinker with the programs and stretch out with your own code.

It is assumed the reader has some familiarity with Fourier optics. Presenting the topic from the ground up is too much material to cover and would obscure our purpose. However, the analytic theory required is presented in summary form throughout the text. The notation and form closely follow Goodman's presentation in *Introduction to Fourier Optics*.[2] For further details and explanations of the analytic foundations of Fourier theory and Fourier optics the reader is encouraged to consult Goodman's book as well as the many other excellent references that exist on the topic.[4–7]

## 1.2 The Realm of Computational Fourier Optics

In this book, the variables, vectors, and arrays in the computer code are defined as much as possible in terms of physical quantities. For example, the coordinates of samples in an array that models a spatial plane are defined in units of meters. Integers for indexing arrays show up only when they can't be avoided. This approach allows a clear connection between the physical world being modeled and the computer code. MATLAB's vectorized structure is also suited to this approach. Thus, programming examples presented in the book involve specific aperture sizes, wavelengths, and distances. Although some examples are simply academic, others are something one might encounter in the real world. However, the reader will soon notice an emergent theme: the finite size of the sample array in the computer limits the range of parameters that can be considered.

We might consider this difficulty in light of the optical designer's dilemma: When does one transition between a geometrical optics prediction of system performance and a wave optics prediction? The difference between these predictors is that geometrical optics assumes rectilinear (straight-line) propagation of the rays of light and ignores diffractive spreading due to the wave nature of light. The usual answer for the dilemma is that for small departures from perfection (near the "diffraction limit") a wave optics description is needed. For large departures a geometrical ray optics description, which has more flexible implementation options, is adequate.[8,9]

So, although analytic Fourier optics theory is quite general, the finite array size tends to limit the computer modeling to the "near-perfection" situations. Typically, this means small divergence angles for optical beam propagation, small simulated image area, and so forth. For practical applications, this is the same realm as the wave optics performance prediction for optical system design.

The remainder of this chapter is a summary of the fundamental Fourier transform definitions, theorems, basic functions, and transform pairs. A review of linear systems theory is also included. So, let's go!

## 1.3 Fourier Transform Definitions and Existence

Fourier optics problems often involve two spatial dimensions. The analytic Fourier transform of a function *g* of two variables *x* and *y* is given by

$$G(f_X, f_Y) = \int_{-\infty}^{\infty}\int g(x,y)\exp\left[-j2\pi(f_X x + f_Y y)\right]dxdy\,, \tag{1.1}$$

where $G(f_X, f_Y)$ is the transform result and $f_X$ and $f_Y$ are independent frequency variables associated with *x* and *y*, respectively. This operation is often described in a shorthand manner as $\Im\{g(x,y)\} = G(f_X, f_Y)$. Similarly, the analytic inverse Fourier transform is given by

$$g(x,y) = \int_{-\infty}^{\infty}\int G(f_X, f_Y)\exp\left[j2\pi(f_X x + f_Y y)\right]df_X df_Y\,. \tag{1.2}$$

The shorthand notation for this operation is $\Im^{-1}\{G(f_X, f_Y)\} = g(x,y)$.

For the Fourier transform to be realizable in a mathematical sense, $g(x,y)$ must satisfy certain sufficient conditions. These conditions are commonly listed as:

(a) *g* must be absolutely integrable over the infinite range of *x* and *y*;

(b) *g* must have only a finite number of discontinuities; and

(c) *g* must have no infinite discontinuities.

Goodman[2] illustrates that in a number of important cases, one or more of these conditions can be weakened, and a generalized transform approach using idealized mathematical functions can be employed to find useful transform representations. Some generalized transform results of interest include

$$\Im\{1\} = \delta(f_X, f_Y)\,,$$

$$\Im\{\cos(2\pi f_0 x)\} = \tfrac{1}{2}\delta(f_X - f_0, f_Y) + \tfrac{1}{2}\delta(f_X + f_0, f_Y)\,,$$

where $\delta$ is the Dirac delta function.

## 1.4 Theorems and Separability

The theorems listed in Table 1.1 find considerable application in Fourier analysis. In Table 1.1, *A*, *B*, *a*, and *b* are scalar constants.

An important property of certain functions is *separability*. A two-dimensional (2D) function is separable if it can be written as the product of two functions of a single variable, such as

$$g_S(x,y) = g_X(x)g_Y(y). \tag{1.3}$$

Separability reduces the Fourier transform of a 2D function to the product of two one-dimensional (1D) transforms or

$$\Im\{g_S(x,y)\} = \Im\{g_X(x)\}\Im\{g_Y(y)\}. \tag{1.4}$$

**Table 1.1** Fourier transform theorems.

| Theorem | Expression |
|---|---|
| Linearity | $\Im\{Ag(x,y)+Bh(x,y)\} = A\Im\{g(x,y)\} + B\Im\{h(x,y)\}$ |
| Similarity | $\Im\left\{g\left(\frac{x}{a},\frac{y}{b}\right)\right\} = \lvert ab\rvert G(af_X, bf_Y)$ |
| Shift | $\Im\{g(x-a,y-b)\} = G(f_X,f_Y)\exp\left[-j2\pi(f_Xa+f_Yb)\right]$ |
| Parseval's (Rayleigh's) | $\iint \lvert g(x,y)\rvert^2 dxdy = \iint \lvert G(f_X,f_Y)\rvert^2 df_X df_Y$ |
| Convolution | $\Im\left\{\iint g(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta\right\} = G(f_X,f_Y)H(f_X,f_Y)$ |
| Autocorrelation | $\Im\left\{\iint g(\xi,\eta)g^*(\xi-x,\eta-y)d\xi d\eta\right\} = \lvert G(f_X,f_Y)\rvert^2$<br>$\Im\left\{\lvert g(x,y)\rvert^2\right\} = \iint G(\xi,\eta)G^*(\xi-f_X,\eta-f_Y)d\xi d\eta$ |
| Cross-correlation | $\Im\left\{\iint g(\xi,\eta)h^*(\xi-x,\eta-y)d\xi d\eta\right\} = G(f_X,f_Y)H^*(f_X,f_Y)$<br>$\Im\{g(x,y)h^*(x,y)\} = \iint G(\xi,\eta)H^*(\xi-f_X,\eta-f_Y)d\xi d\eta$ |
| Fourier integral | $\Im\Im^{-1}\{g(x,y)\} = \Im^{-1}\Im\{g(x,y)\} = g(x,y)$ |
| Successive transform | $\Im\Im\{g(x,y)\} = g(-x,-y)$ |
| Central ordinate | $\Im\{g(x,y)\}\big\rvert_{f_X=0,\, f_Y=0} = G(0,0) = \int\int g(x,y)dxdy$<br>$\Im^{-1}\{G(f_X,f_Y)\}\big\rvert_{x=0,\, y=0} = g(0,0) = \int\int G(f_X,f_Y)df_X df_Y$ |

Note: $A$, $B$, $a$, and $b$ are scalar constants

## 1.5 Basic Functions and Transforms

Several basic functions, or combinations thereof, are used to describe various physical or analytic structures encountered in optics, such as a circle function to describe a circular aperture. Thus, these functions and their Fourier transform pairs are of considerable utility. The definitions in Table 1.2 are adopted.

Functions of one variable are illustrated in Fig. 1.1. These can be combined as products to represent separable 2D functions. The circle function is a symmetric 2D function where a single radial variable $r = (x^2+y^2)^{1/2}$ is often used. A shorthand name is not defined for the Gaussian, but this function appears often. The form we use is convenient for Fourier analysis. The circle and a 2D Gaussian function are plotted in Fig. 1.2 for illustration.

**Table 1.2** Basic functions.

| Function | Definition |
|---|---|
| Rectangle | $\mathrm{rect}(x) = \begin{cases} 1, & \lvert x\rvert < \frac{1}{2} \\ \frac{1}{2}, & \lvert x\rvert = \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$ |
| Sinc | $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$ |
| Triangle | $\Lambda(x) = \begin{cases} 1-\lvert x\rvert, & \lvert x\rvert \le 1, \\ 0, & \text{otherwise.} \end{cases}$ |
| Comb | $\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$ |
| Gaussian | $\exp\left(-\pi x^2\right)$ |
| Circle | $\mathrm{circ}\left(\sqrt{x^2+y^2}\right) = \begin{cases} 1 & \sqrt{x^2+y^2} < \frac{1}{2}, \\ \frac{1}{2} & \sqrt{x^2+y^2} = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$ |