1 Mathematical Preliminaries

We shall go through in this first chapter all of the mathematics needed for reading the rest of this book.
The reader is expected to have taken a one-year course in differential and integral calculus.

1.1 Mean-Value Theorems of Integral Calculus

First mean-value theorem of integral calculus
Let $f(x)$ be continuous on $[a,b]$ and $g(x) > 0$ (or $g(x) < 0$) in $[a,b]$.
Then,

$$\int_a^b f(x)g(x)dx = f(x_1)\int_a^b g(x)dx,$$

where x_1 is in $[a,b]$.
Proof
We shall prove the case for $g(x) > 0$; the case for $g(x) < 0$ is entirely analogous.
Since $f(x)$ is continuous on $[a,b]$, it is bounded, i.e., there exist m and M such that $m \leq f(x) \leq M$ for all x in $[a,b]$.
We further have $mg(x) \leq f(x)g(x) \leq Mg(x)$ for all x in $[a,b]$ since $g(x) > 0$ for all x in $[a,b]$.
Hence,

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$$

$$m \leq \frac{1}{I_g}\int_a^b f(x)g(x)dx \leq M; \qquad I_g = \int_a^b g(x)dx. \tag{1}$$

Since $f(x)$ is continuous on $[a,b]$, it must evolve continuously between m and M.
Hence, for any y_1 satisfying $m \leq y_1 \leq M$, there exists an x_1 in $[a,b]$ such that $f(x_1) = y_1$.
Now, apply the above statement to (1).
Let

$$\frac{1}{I_g}\int_a^b f(x)g(x)dx = y_1.$$

Since $m \le y_1 \le M$, there exists x_1 in $[a, b]$ such that $f(x_1) = y_1$.
Hence,

$$\frac{1}{I_g}\int_a^b f(x)g(x)dx = f(x_1).$$

That is,

$$\int_a^b f(x)g(x)dx = f(x_1)\int_a^b g(x)dx.$$

■

In particular, if $g(x) = 1$, we have

$$\int_a^b f(x)dx = f(x_1)\int_a^b dx = f(x_1)(b-a).$$

Second mean-value theorem of integral calculus

Let $f(x)$ be monotonically increasing (or decreasing) on $[a, b]$ and $g(x)$ be integrable on $[a, b]$. Then,

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{x_1} g(x)dx + f(b)\int_{x_1}^b g(x)dx,$$

where x_1 is in $[a, b]$.

Proof

We assume first that $f(x)$ is monotonically increasing, implying that $f'(x) > 0$ on $[a, b]$.
Let

$$G(x) = \int^x g(x_\wedge)dx_\wedge + c.$$

Hence, G is differentiable and thus continuous on $[a, b]$.

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x)$$

$$= f(x)G(x)|_a^b - \int_a^b G(x)f'(x)dx$$

By first mean-value theorem

$$= f(b)G(b) - f(a)G(a) - G(x_1)[f(b) - f(a)]$$
$$= f(a)[G(x_1) - G(a)] + f(b)[G(b) - G(x_1)]$$

$$= f(a)\int_{a}^{x_1} g(x)dx + f(b)\int_{x_1}^{b} g(x)dx.$$

■

1.2 The Delta Function

Definition

We define the delta function, denoted conventionally as $\delta(x)$, to be the limit of a sequence of functions in the sense that, if

$$\lim_{n\to\infty}\int_{0^+}^{b} D_n(x)dx = \frac{1}{2}, \qquad b > 0$$

or

$$\lim_{n\to\infty}\int_{b}^{0^-} D_n(x)dx = \frac{1}{2}, \qquad b < 0,$$

then

$$\lim_{n\to\infty} D_n(x) = \delta(x).$$

Claim

$$\lim_{n\to\infty}\int_{c}^{d} D_n(x)dx = 0,$$

where $0 < c < d$ or $c < d < 0$.

Proof

For $0 < c < d$,

$$\lim_{n\to\infty}\int_{c}^{d} D_n(x)dx = \lim_{n\to\infty}\int_{0^+}^{d} D_n(x)dx - \lim_{n\to\infty}\int_{0^+}^{c} D_n(x)dx = \frac{1}{2}-\frac{1}{2} = 0.$$

The case for $c < d < 0$ can be proved in a similar way.

■

1.2.1 Representations of the delta function

1.)

One representation of the delta function is

$$\frac{\sin(\beta x)}{\pi x}, \quad \beta \longrightarrow \infty.$$

We use β instead of n as the index of the sequence of functions since it is not restricted to integers.

This is because

$$\lim_{\beta\to\infty} \int_{0^+}^{b} \frac{\sin(\beta x)}{\pi x} dx = \lim_{\beta\to\infty} \frac{1}{\pi} \int_{0^+}^{\beta b} \frac{\sin x_\wedge}{x_\wedge} dx_\wedge = \frac{1}{\pi} \int_{0^+}^{\infty} \frac{\sin x_\wedge}{x_\wedge} dx_\wedge = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}.$$

The evaluation of the last integral is detailed in Appendix 1.1.

To show the reader this trend, we plot the representation for β from 1 (blue) to 5 (red) in steps of 1, as shown in Fig. 1.2-1.

2.)

Another representation of the delta function is

$$\frac{\beta e^{-\beta^2 x^2}}{\sqrt{\pi}}, \quad \beta \longrightarrow \infty.$$

This is because

$$\lim_{\beta\to\infty} \int_{0^+}^{b} \frac{\beta e^{-\beta^2 x^2}}{\sqrt{\pi}} dx = \lim_{\beta\to\infty} \frac{1}{\sqrt{\pi}} \int_{0^+}^{\beta b} e^{-x_\wedge^2} dx_\wedge = \frac{1}{\sqrt{\pi}} \int_{0^+}^{\infty} e^{-x_\wedge^2} dx_\wedge = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = \frac{1}{2}.$$

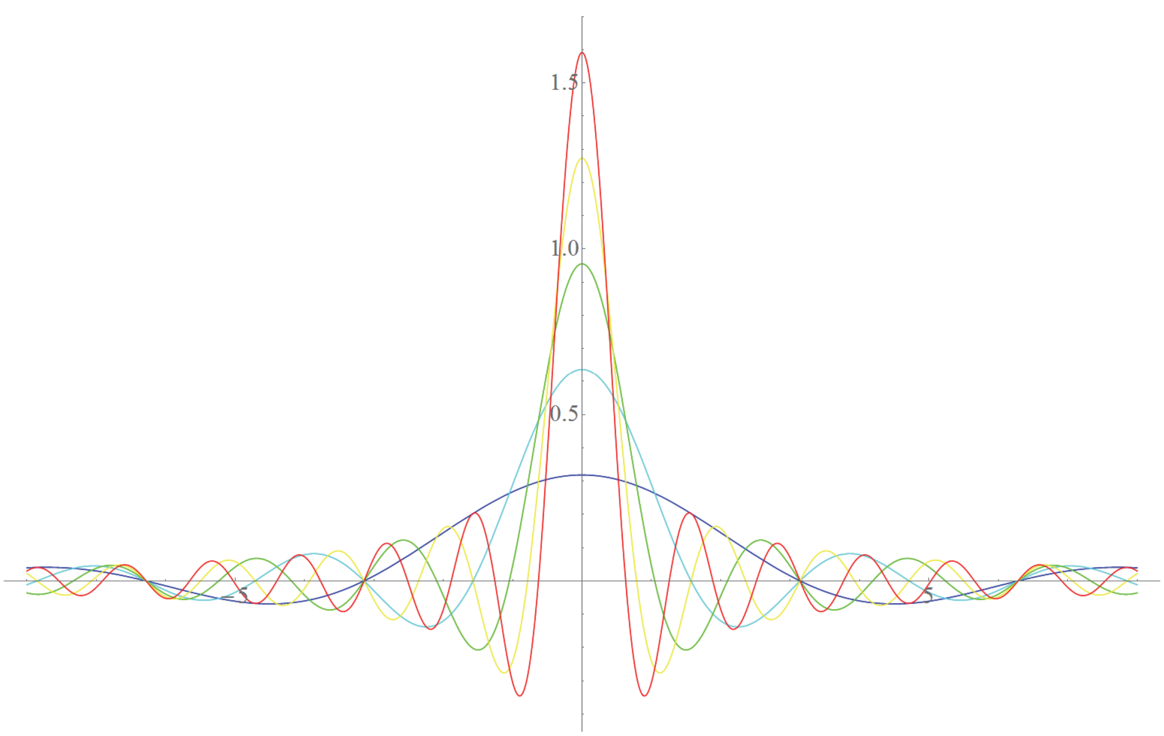

Figure 1.2-1

3.)

Still another representation of the delta function is

$$\frac{1}{2\pi}\int_{-\beta}^{\beta} e^{ikx}dk\,, \qquad \beta \longrightarrow \infty.$$

This is because, by performing the integration

$$\frac{1}{2\pi}\int_{-\beta}^{\beta} e^{ikx}dk = \frac{1}{2\pi}\frac{e^{i\beta x}-e^{-i\beta x}}{ix} = \frac{\sin(\beta x)}{\pi x},$$

we can reduce it to the first representation above.

4.)

Our final example of the representation of the delta function is the following sequence of polynomials:

$p_n(x) = a_n(1-x^2)^n, \quad |x| \le 1; \quad p_n(x) = 0, \quad |x| > 1, \quad n = 1, 2, 3, \cdots,$

where a_n is a normalization factor defined by

$$a_n \int_{0^+}^{1} (1-x^2)^n dx = \frac{1}{2}, \qquad n = 1, 2, 3, \cdots.$$

Proof

When $0 < b < 1$, consider the following integral

$$\int_{0^+}^{b} p_n(x)dx = a_n \int_{0^+}^{b} (1-x^2)^n dx$$

$$= a_n \int_{0^+}^{1} (1-x^2)^n dx - a_n \int_{b}^{1} (1-x^2)^n dx$$

$$= \frac{1}{2} - a_n \int_{b}^{1} (1-x^2)^n dx.$$

On one hand,

$$\frac{1}{a_n} \equiv 2\int_{0^+}^{1} (1-x^2)^n dx > 2\int_{0^+}^{1} (1-x)^n dx = 2\frac{-(1-x)^{n+1}}{n+1}\bigg|_{0^+}^{1} = \frac{2}{n+1};$$

$$a_n < \frac{n+1}{2}.$$

On the other hand,

$$\int_b^1 (1-x^2)^n dx < (1-b^2)^n(1-b) < (1-b^2)^n.$$

Hence,

$$a_n \int_b^1 (1-x^2)^n dx < \frac{n+1}{2}(1-b^2)^n \longrightarrow 0, \quad n \longrightarrow \infty.$$

Then,

$$\int_{0^+}^b p_n(x)dx = \frac{1}{2}.$$

When $b \geq 1$,

$$\int_{0^+}^b p_n(x)dx = \int_{0^+}^1 p_n(x)dx = a_n \int_{0^+}^1 (1-x^2)^n dx = \frac{1}{2}$$

by the definition of $p_n(x)$ and a_n.
Therefore,

$$\lim_{n\longrightarrow\infty} \int_{0^+}^b a_n(1-x^2)^n dx = \frac{1}{2}, \quad b > 0,$$

which means $p_n(x), \quad n \longrightarrow \infty$ is indeed a representation of the δ function.
■

1.2.2 Properties of the delta function

1.)
Sifting
When $b > 0$, if $f(x)$ is continuous on $(0, b]$ and $f(x)|_{x\longrightarrow 0^+} = f(0^+)$, then

$$\int_{0^+}^b f(x)\delta(x)dx = \frac{1}{2}f(0^+).$$

Similarly, when $b < 0$, if $f(x)$ is continuous on $[b, 0)$ and $f(x)|_{x\longrightarrow 0^-} = f(0^-)$, then

$$\int_b^{0^-} f(x)\delta(x)dx = \frac{1}{2}f(0^-).$$

In particular,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \frac{1}{2}[f(0^-) + f(0^+)],$$

which is equal to $f(0)$ if $f(x)$ is continuous at $x = 0$.

Proof

We first prove the case of $b > 0$.

Since $f(x)$ is continuous, there exists a small enough b_1 such that $f(x)$ is monotonic on $(0, b_1]$.

We then apply the second mean-value theorem of integral calculus and get

$$\int_{0^+}^{b_1} f(x)\delta(x)dx = \lim_{n\to\infty} \int_{0^+}^{b_1} f(x)D_n(x)dx$$

$$= \lim_{n\to\infty} f(0^+) \int_{0^+}^{x_1} D_n(x)dx + \lim_{n\to\infty} f(b_1) \int_{x_1}^{b_1} D_n(x)dx$$

$$= f(0^+) * \frac{1}{2} + f(b_1) * 0.$$

For $b > 0$ in general, we write

$$\int_{0^+}^{b} f(x)\delta(x)dx = \int_{0^+}^{b_1} f(x)\delta(x)dx + \int_{b_1}^{b} f(x)\delta(x)dx = \frac{1}{2}f(0^+) + \int_{b_1}^{b} f(x)\delta(x)dx.$$

Next, we divide $[b_1, b]$ into several sub-intervals in which $f(x)$ is monotonic.

Let one such sub-interval be $[b_{k-1}, b_k]$.

Then,

$$\int_{b_{k-1}}^{b_k} f(x)\delta(x)dx = \lim_{n\to\infty} \int_{b_{k-1}}^{b_k} f(x)D_n(x)dx$$

$$= \lim_{n\to\infty} f(b_{k-1}) \int_{b_{k-1}}^{x_k} D_n(x)dx + \lim_{n\to\infty} f(b_k) \int_{x_k}^{b_k} D_n(x)dx$$

$$= f(b_{k-1}) * 0 + f(b_k) * 0.$$

Hence, adding up all such integrals, we have

$$\int_{b_1}^{b} f(x)\delta(x)dx = 0.$$

Therefore, when $b > 0$,

$$\int_{0^+}^{b} f(x)\delta(x)dx = \frac{1}{2}f(0^+).$$

Similarly, when $b < 0$,

$$\int_{b}^{0^-} f(x)\delta(x)dx = \frac{1}{2}f(0^-).$$

If $f(x)$ is continuous at $x = 0$,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \frac{1}{2}[f(0^-) + f(0^+)] = f(0).$$

■

Letting in the above equation $f(x) = g(x + c)$, we have

$$\int_{-\infty}^{\infty} g(x + c)\delta(x)dx = g(c).$$

By change of variable $x + c = x_\wedge$, we obtain

$$\int_{-\infty}^{\infty} g(x_\wedge)\delta(x_\wedge - c)dx_\wedge = g(c).$$

The above expression is the most common form for expressing the sifting property of the delta function.

2.)

Scaling

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{|a|}f(0).$$

Proof

If $a > 0$,

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \int_{-\infty}^{\infty} f(x_\wedge/a)\delta(x_\wedge)\frac{dx_\wedge}{a}$$

$$= \frac{1}{a}f(0).$$

If $a < 0$,

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \int_{\infty}^{-\infty} f(x_\wedge/a)\delta(x_\wedge)\frac{dx_\wedge}{a}$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} f(x_\wedge/a)\delta(x_\wedge)dx_\wedge$$

$$= -\frac{1}{a}f(0).$$

■

3.)

Functional

$$\delta[g(x)] = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i).$$

Proof

$\delta(x)$ is non-trivial only in the neighborhood of $x = 0$.

Thus, for $\delta[g(x)]$, we can only focus on those tiny intervals centered at x_i's where $g(x_i) = 0$, and on each such interval, approximate $g(x)$ by a linear function, i.e.,

$$g(x) \approx g(x_i) + g'(x_i)(x - x_i).$$

Hence,

$$\int_{-\infty}^{\infty} f(x)\delta[g(x)]dx = \sum_i \int_{-\infty}^{\infty} f(x)\delta[g'(x_i)(x - x_i)]dx.$$

$$= \sum_i \frac{1}{|g'(x_i)|} f(x_i).$$

We may then state, equivalently,

$$\delta[g(x)] = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i).$$

■

4.)

Differentiation

$$\int_{-\infty}^{\infty} f(x)\delta'(x - c)dx = -f'(c).$$

Proof

$$\int_{-\infty}^{\infty} f(x)\delta'(x - c)dx = \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(x)\frac{\delta(x + \Delta/2 - c) - \delta(x - \Delta/2 - c)}{\Delta}dx$$

$$= \lim_{\Delta \to 0} \frac{f(c - \Delta/2) - f(c + \Delta/2)}{\Delta}$$

$$= -\lim_{\Delta \to 0} \frac{f(c + \Delta/2) - f(c - \Delta/2)}{\Delta}$$

$$= -f'(c).$$

1.3 Weierstrass' Approximation Theorem

Weierstrass' approximation theorem states that any function which is continuous in an interval can be approximated uniformly by polynomials, i.e., $1,\ x,\ x^2, \ldots$, in this interval.

Weierstrass' approximation theorem can be explained by employing the sifting property of the delta function we have just proved.
Assume that $f(x)$ is continuous in $[c,d]$.
Then,

$$f(x) = \int_{c^-}^{d^+} f(u)\delta(u-x)du$$

$$= \lim_{n\to\infty} a_n \int_{c^-}^{d^+} f(u)[1-(u-x)^2]^n du,$$

by employing the polynomial representation of the delta function.
Here, $c^- < c < d < d^+$.
Why is the integration domain $[c^-, d^+]$ larger than $[c,d]$?
If we integrate over $[c,d]$, then at the boundary, e.g., at c, we only get $f(c)/2$, not $f(c)$.
After performing the integration in the above equation, we obtain a polynomial of order $2n$.
We need to choose a proper n to meet the required error tolerance.

For an explicit proof of Weierstrass' approximation theorem, see Appendix 1.2.

1.4 Fourier Transform

We define the Fourier transform as

$$\mathcal{F}[U(x)] \equiv \int_{-\infty}^{\infty} U(x)e^{-ikx}dx = \widetilde{U}(k)$$

and the inverse Fourier transform as

$$\mathcal{F}^{-1}\left[\widetilde{U}(k)\right] \equiv \int_{-\infty}^{\infty} \widetilde{U}(k)e^{ikx}\frac{dk}{2\pi}.$$

$U(x)$ and $\widetilde{U}(k)$ are called Fourier transform pairs.
The functions $U(x)$ and $\widetilde{U}(k)$ are generally complex; however, the variables x and k are always real unless otherwise stated.

1.4.1 Fourier transform theorems

1.)

Fourier integral theorem

$\mathcal{F}^{-1}\big[\mathcal{F}[U(x)]\big] = U(x).$

Proof

$$\mathcal{F}^{-1}\big[\mathcal{F}[U(x)]\big] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dx_1 e^{-ikx_1} U(x_1)$$

$$= \int_{-\infty}^{\infty} dx_1 U(x_1) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik(x_1-x)}$$

$$= \int_{-\infty}^{\infty} U(x_1)\delta(x_1 - x) dx$$

$= U(x).$

If $U(x)$ is discontinuous at x, replace $\delta(x_1 - x)$ by $D_n(x_1 - x)$ in the second-to-last equation and let $n \longrightarrow \infty$.

We see that the newly obtained $U(x)$ is the average of $U(x)$ in the neighborhood of x.

■

2.)

Linearity theorem

$\mathcal{F}[a_1 U_1(x) + a_2 U_2(x)] = a_1 \widetilde{U}_1(k) + a_2 \widetilde{U}_2(k).$

3.)

Scaling theorem

$$\mathcal{F}[U(ax)] = \frac{1}{|a|} \widetilde{U}(k/a).$$

Proof

If $a > 0$,

$$\mathcal{F}[U(ax)] = \int_{-\infty}^{\infty} U(ax) e^{-ikx} dx$$

$ax = x_1$

$$= \int_{-\infty}^{\infty} U(x_1) e^{-ik\frac{x_1}{a}} \frac{dx_1}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} U(x_1)e^{-i\frac{k}{a}x_1}dx_1$$

$$= \frac{1}{a}\widetilde{U}(k/a).$$

If $a < 0$,

$$\mathcal{F}[U(ax)] = \int_{-\infty}^{\infty} U(ax)e^{-ikx}dx$$

$ax = x_1$

$$= \int_{\infty}^{-\infty} U(x_1)e^{-ik\frac{x_1}{a}}\frac{dx_1}{a}$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} U(x_1)e^{-i\frac{k}{a}x_1}dx_1$$

$$= -\frac{1}{a}\widetilde{U}(k/a).$$

■

4.)

Shift theorem

$\mathcal{F}[U(x-c)] = e^{-ikc}\widetilde{U}(k).$

Proof

$$\mathcal{F}[U(x-c)] = \int_{-\infty}^{\infty} U(x-c)e^{-ikx}dx$$

$$= e^{-ikc}\int_{-\infty}^{\infty} U(x-c)e^{-ik(x-c)}d(x-c)$$

$x - c = x_1$

$$= e^{-ikc}\int_{-\infty}^{\infty} U(x_1)e^{-ikx_1}dx_1$$

$$= e^{-ikc}\widetilde{U}(k).$$

■

5.)

Rayleigh's (Parseval's) theorem

$$\int_{-\infty}^{\infty} |U(x)|^2 dx = \int_{-\infty}^{\infty} \left|\tilde{U}(k)\right|^2 \frac{dk}{2\pi}.$$

Proof

$$\int_{-\infty}^{\infty} |U(x)|^2 dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}\tilde{U}(k) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} e^{-ik_1x}\tilde{U}^*(k_1)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\tilde{U}(k) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\tilde{U}^*(k_1) \int_{-\infty}^{\infty} dx e^{-i(k_1-k)x}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\tilde{U}(k) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\tilde{U}^*(k_1) 2\pi\delta(k_1-k)$$

$$= \int_{-\infty}^{\infty} \left|\tilde{U}(k)\right|^2 \frac{dk}{2\pi}.$$

■

6.)

Convolution theorem

The convolution of two functions is defined as

$$U(x)\otimes V(x) \equiv \int_{-\infty}^{\infty} U(x-x_1)V(x_1)dx_1$$

$x - x_1 = x_2$

$$= \int_{\infty}^{-\infty} V(x-x_2)U(x_2)(-dx_2)$$

$$= \int_{-\infty}^{\infty} V(x-x_2)U(x_2)dx_2$$

$= V(x)\otimes U(x).$

Then,

$$\mathcal{F}\{U(x)\otimes V(x)\} = \int_{-\infty}^{\infty} dx e^{-ikx} \int_{-\infty}^{\infty} dx_1 U(x-x_1)V(x_1)$$

$$= \int_{-\infty}^{\infty} dx_{\wedge} e^{-ikx_1} V(x_1) \int_{-\infty}^{\infty} d(x - x_1) e^{-ik(x-x_1)} U(x - x_1)$$

$$= \tilde{V}(k)\tilde{U}(k).$$

Besides,

$$\mathcal{F}[U(x)V(x)] = \int_{-\infty}^{\infty} U(x)V(x)e^{-ikx}dx$$

$$= \int_{-\infty}^{\infty} dxe^{-ikx} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} e^{ik_1x}\tilde{U}(k_1) \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{ik_2x}\tilde{V}(k_2)$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \tilde{U}(k_1) \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tilde{V}(k_2) \int_{-\infty}^{\infty} dxe^{-i(k-k_1-k_2)x}$$

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \tilde{U}(k_1) \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tilde{V}(k_2) 2\pi\delta(k - k_1 - k_2)$$

either

$$= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \tilde{V}(k - k_1)\tilde{U}(k_1)$$

$$= \tilde{V}(k) \otimes \tilde{U}(k)$$

or

$$= \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tilde{U}(k - k_2)\tilde{V}(k_2).$$

$$= \tilde{U}(k) \otimes \tilde{V}(k).$$

7.)

Complex conjugate

$$\mathcal{F}[U^*(x)] = \int_{-\infty}^{\infty} U^*(x)e^{-ikx}dx$$

$$= \left[\int_{-\infty}^{\infty} U(x)e^{ikx}dx\right]^*$$

$x = -x_1$

$$= \left[\int_{\infty}^{-\infty} U(-x_1)e^{-ikx_1}(-dx_1)\right]^*$$

$$= \left[\int_{-\infty}^{\infty} U(-x_1)e^{-ikx_1}dx_1\right]^*$$

$$= \left[\mathcal{F}[U(-x)]\right]^*$$

$$\neq \left[\mathcal{F}[U(x)]\right]^*.$$

That is, the operations of the Fourier transform and complex conjugate do not commute unless $U(-x) = U(x)$, i.e., for functions with inversion symmetry.

8.)

Autocorrelation theorem

$$\mathcal{F}[U(x) \otimes U^*(-x)] = \mathcal{F}[U(x)]\left[\mathcal{F}[U(x)]\right]^* = \widetilde{U}(k)\widetilde{U}^*(k) = \left|\widetilde{U}(k)\right|^2.$$

In addition,

$$\mathcal{F}[|U(x)|^2] = \mathcal{F}[U(x)U^*(x)] = \mathcal{F}[U(x)] \otimes \mathcal{F}[U^*(x)] = \widetilde{U}(k) \otimes \widetilde{U}^*(-k).$$

1.4.2 Useful Fourier transform pairs

1.)

Fourier transform of the rectangle function

The rectangle function in the real space of width W_x is defined as

$$\mathrm{Rect}(x/W_x) = \begin{cases} 1, & |x| < W_x/2 \\ 1/2, & |x| = W_x/2 \\ 0, & |x| > W_x/2\,. \end{cases}$$

Finding its Fourier transform is straightforward:

$$\mathcal{F}[\mathrm{Rect}(x/W_x)] = \int_{-W_x/2}^{W_x/2} e^{-ikx}dx$$

$$= \left.\frac{e^{-ikx}}{-ik}\right|_{x=-W_x/2}^{W_x/2}$$

$$= \frac{e^{-ikW_x/2} - e^{ikW_x/2}}{-ik}$$

$$= \frac{-i2\sin(kW_x/2)}{-ik}$$

$= W_x \frac{\sin(kW_x/2)}{kW_x/2}$

$= W_x \mathrm{Sinc}(kW_x/2).$

The rectangular function in the frequency space may be employed more frequently.

Similarly, it can be shown that

$$\mathcal{F}^{-1}[\mathrm{Rect}(k/W_k)] = \frac{W_k}{2\pi}\mathrm{Sinc}(W_k x/2).$$

2.)

Fourier transform of the comb function

The comb function is defined as

$$\delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np),$$

which is a periodic function of period p.

We want to compute its Fourier transform.

$$\tilde{\delta}_p(k) = \int_{-\infty}^{\infty} \delta_p(x) e^{-ikx} dx$$

$$= \sum_{n=-\infty}^{\infty} e^{-i*k*np}$$

$$= \sum_{n=-\infty}^{\infty} e^{i*k*np}$$

e^{i*k*p} is a periodic function of period $2\pi/p$.

e^{i*k*2p} is a periodic function of period $2\pi/2p$, which is also a periodic function of period $2\pi/p$.

...

Therefore, $\tilde{\delta}_p(k)$ is also a periodic function of period $2\pi/p$.

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} \left[e^{ikp}\right]^n$$

$$= \lim_{N\to\infty} \left[e^{ikp}\right]^{-N} \frac{\left[e^{ikp}\right]^{2N+1} - 1}{e^{ikp} - 1}$$

$$= \lim_{N\to\infty} \frac{\left[e^{ikp}\right]^{N+1} - \left[e^{ikp}\right]^{-N}}{e^{ikp} - 1}$$

$$\left[e^{ikp}\right]^{N+1} - \left[e^{ikp}\right]^{-N} = \left[e^{ikp}\right]^{1/2}\left[\left[e^{ikp}\right]^{N+1/2} - \left[e^{ikp}\right]^{-(N+1/2)}\right]$$

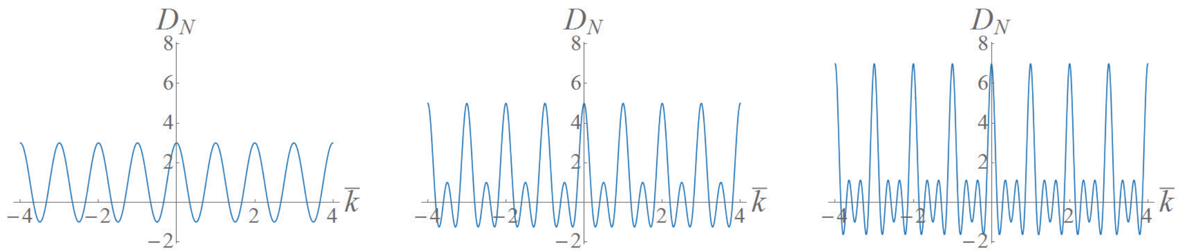

Figure 1.4-1

$e^{ikp} - 1 = [e^{ikp}]^{1/2}\left[[e^{ikp}]^{1/2} - [e^{ikp}]^{-1/2}\right]$

$$= \lim_{N\to\infty} \frac{\sin[(N+1/2)kp]}{\sin(kp/2)}$$

$$\equiv \lim_{N\to\infty} D_N(k).$$

$D_N(k)$ may diverge at $k = m(2\pi/p)$ since its denominator equals zero there.

Before going into the mathematical details, we first plot, in Fig. 1.4-1, $D_N(k)$ versus k for $N = 1$, 2, 3.
(Actually, we plot $D_N(k)$ versus $\bar{k}$, defined by $k = (2\pi/p)\bar{k}$.)

It is seen that as N increases, the main lobes at $k = m(2\pi/p)$ become higher (though narrower), whereas the side lobes become lower (if normalized by the main lobe at $k = 0$).
Yes, your guess is correct.
It is an infinite series of delta functions.
We sketch a formal proof below.

Proof

First, we consider

$$\int_{-\pi/p}^{\pi/p} f(k)D_N(k)dk = \int_{-\pi/p}^{\pi/p} f(k)\frac{\sin[(N+1/2)kp]}{\sin(kp/2)}dk$$

$(N+1/2)kp = v$

$$= \int_{-(N+1/2)\pi}^{(N+1/2)\pi} f[v/(N+1/2)p]\frac{\sin v}{\sin[v/2(N+1/2)]}\frac{dv}{(N+1/2)p}$$

$$= \frac{2}{p}\int_{-(N+1/2)\pi}^{(N+1/2)\pi} f[v/(N+1/2)p]\frac{\sin v/v}{\sin[v/2(N+1/2)]/[v/2(N+1/2)]}dv$$

$$\lim_{N\to\infty}\frac{\sin[v/2(N+1/2)]}{v/2(N+1/2)}=\lim_{v_1\to 0}\frac{\sin v_1}{v_1}=1$$

$$\to\frac{2}{p}\int_{-\infty}^{\infty} f(0)\frac{\sin v}{v}dv\,,\qquad N\to\infty$$

$$=\frac{4}{p}f(0)\int_{0}^{\infty}\frac{\sin v}{v}dv$$

$$\int_{0^+}^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$$

$$=\frac{2\pi}{p}f(0).$$

Since $D_N(k)$ is a periodic function of period $2\pi/p$,

$$\tilde{\delta}_p(k)=\lim_{N\to\infty}D_N(k)=\frac{2\pi}{p}\sum_{m=-\infty}^{\infty}\delta[k-m(2\pi/p)].$$

■

Computing the inverse Fourier transform of the above equation, we obtain

$$\delta_p(x)=\int_{-\infty}^{\infty}\tilde{\delta}_p(k)e^{ikx}\frac{dk}{2\pi}$$

$$=\frac{1}{p}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta[k-m(2\pi/p)]e^{ikx}dk$$

$$=\frac{1}{p}\sum_{m=-\infty}^{\infty}e^{i*m\frac{2\pi}{p}*x}.$$

In summary,

$$\delta_p(x)=\sum_{n=-\infty}^{\infty}\delta(x-np)=\frac{1}{p}\sum_{m=-\infty}^{\infty}e^{i*m\frac{2\pi}{p}*x};$$

$$\tilde{\delta}_p(k)=\sum_{n=-\infty}^{\infty}e^{i*k*np}=\frac{2\pi}{p}\sum_{m=-\infty}^{\infty}\delta[k-m(2\pi/p)]. \qquad (1)$$