Chapter 4

Two Standard Shortcuts Used to Transform Electromagnetic Equations

The last several chapters have explained how the standard rules for changing units apply to electromagnetic physical quantities. Having become familiar with these rules, we are now sure that electromagnetic equations and formulas transform in a way that makes sense when going from one system of units to another. We also know, however, that following these rules can be algebraically cumbersome, forcing us always to watch for the appearance or disappearance of constants ε_0 and μ_0 as we recognize or refuse to recognize charge as a new dimension. Engineers and physicists are no more eager than anyone else to do unnecessary work; consequently, they have come up with both the free-parameter method and substitution tables, two shortcuts that can greatly reduce the time required to convert electromagnetic equations and formulas from one system of units to another. Unfortunately, neither shortcut is perfect: substitution tables can give ambiguous answers in unusual situations, and to apply the free-parameter method we must first relate our equation or formula to one or more of a predefined list of equations and formulas. Nevertheless, these shortcuts often provide a quick and easy way of transforming electromagnetic expressions; and whenever there is any doubt about the result, the transformation can be checked using the procedures explained in the previous chapters.

4.1 The free-parameter method

Table 4.1 lists Maxwell's equations and the Lorentz force law for the six major electromagnetic systems discussed in this book. As pointed out at the beginning of Chapter 3, any classical electromagnetic formula can be derived from Maxwell's equations and the Lorentz force law. This means we can consult Table 4.1, select the appropriate equations in the desired set of units, and from them derive the formulas we need to know. Although this process gets the job done, it usually requires a lot of work. To avoid the unpleasant prospect of deriving all of our formulas and equations from Maxwell's equations and the Lorentz force law, we construct instead a long list of basic electromagnetic equations that contains everything (including Maxwell's equations and the Lorentz force law) likely to be useful. Instead of providing six long lists—one for every electromagnetic system—we use the four free parameters $\widetilde{\varepsilon}$, $\widetilde{\mu}$, k_0, and Π shown in Table 4.2 to reduce the six lists to one.[1]

As an example of how this works, consider what happens when we disregard the "*h*" and "*f*" prefixes and write Maxwell's equations as

$$\vec{\nabla} \cdot \vec{D} = \Pi \rho_Q, \tag{4.1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{4.1b}$$

$$\vec{\nabla} \times \vec{H} = k_0 \left(\Pi \vec{J} + \frac{\partial \vec{D}}{\partial t} \right), \tag{4.1c}$$

$$\vec{\nabla} \times \vec{E} + k_0 \frac{\partial \vec{B}}{\partial t} = 0, \tag{4.1d}$$

where

$$\vec{D} = \widetilde{\varepsilon} \vec{E} + \Pi \vec{P} \tag{4.1e}$$

and

$$\begin{aligned} \vec{H} &= \frac{1}{\widetilde{\mu}} \vec{B} - \frac{\Pi}{\widetilde{\mu}} \vec{M}_H \\ &= \frac{1}{\widetilde{\mu}} \vec{B} - \Pi \vec{M}_I . \end{aligned} \tag{4.1f}$$

As always, $\vec{E}$ and $\vec{D}$ are the electric field and electric displacement, respectively; $\vec{H}$ and $\vec{B}$ are the magnetic field and magnetic induction, respectively; ρ_Q is the volume charge density; $\vec{J}$ is the volume current density; $\vec{P}$ is the electric dipole density; $\vec{M}_H$ is the permanent-magnet dipole density; and $\vec{M}_I$ is the current-loop magnetic dipole density. Clearly, Eqs. (4.1a–f) reduce to the correct set of equations in Table 4.1 when $\widetilde{\varepsilon}$, $\widetilde{\mu}$, k_0, and Π are given the appropriate values from Table 4.2. The same thing can be done to the Lorentz force law; if it is written as

$$\vec{F} = Q\vec{E} + k_0 Q \left(\vec{v} \times \vec{B} \right), \tag{4.2}$$

then it too reduces to the correct equation in Table 4.1 when k_0 is given the appropriate value from Table 4.2.

As we have just seen, the free-parameter method works most easily and naturally when we neglect the distinction between rationalized and unrationalized electromagnetic quantities—that is, neglect the "*h*" and "*f*" prefixes —which so far we have been careful to preserve. It should be emphasized that the distinction between a change of units and a rescaling of an electromagnetic physical quantity is just as important as before; the free-parameter method just makes it inconvenient to keep track of this distinction using a single table. If we want to preserve the distinction between rationalized and unrationalized physical quantities, we can consult Tables 4.3(a) or 4.3(b) after putting an equation or formula into the rationalized mks* or Heaviside-Lorentz systems, respectively.

* As pointed out in Section 3.6 of Chapter 3, most textbooks written today using the rationalized mks system say that they are using SI units.

Table 4.1 Maxwell's equations and the Lorentz force law in the rationalized mks system (which is also called SI units), the unrationalized mks system, Gaussian cgs units, the Heaviside-Lorentz cgs system, esu units, and emu units.

rationalized mks system, also called SI units	$\vec{\nabla}\cdot{}_f\vec{D}=\rho_Q,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,$ $\vec{\nabla}\times{}_f\vec{H}=\vec{J}+\frac{\partial_f\vec{D}}{\partial t},\quad {}_f\vec{D}={}_f\varepsilon_0\vec{E}+\vec{P},$ ${}_f\vec{H}=\frac{\vec{B}}{{}_f\mu_0}-\vec{M}_I,\quad \vec{F}=Q\vec{E}+Q(\vec{v}\times\vec{B})$
unrationalized mks system	$\vec{\nabla}\cdot\vec{D}=4\pi\rho_Q,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,$ $\vec{\nabla}\times\vec{H}=4\pi\vec{J}+\frac{\partial\vec{D}}{\partial t},\quad \vec{D}=\varepsilon_0\vec{E}+4\pi\vec{P},$ $\vec{H}=\frac{\vec{B}}{\mu_0}-4\pi\vec{M}_I,\quad \vec{F}=Q\vec{E}+Q(\vec{v}\times\vec{B})$
Gaussian cgs units	$\vec{\nabla}\cdot\vec{D}=4\pi\rho_Q,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0,$ $\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial\vec{D}}{\partial t},\quad \vec{D}=\vec{E}+4\pi\vec{P},$ $\vec{H}=\vec{B}-4\pi\vec{M}_I,\quad \vec{F}=Q\vec{E}+\frac{Q}{c}(\vec{v}\times\vec{B})$
Heaviside-Lorentz cgs system	$\vec{\nabla}\cdot{}_h\vec{D}={}_h\rho_Q,\quad \vec{\nabla}\cdot{}_h\vec{B}=0,\quad \vec{\nabla}\times{}_h\vec{E}+\frac{1}{c}\frac{\partial_h\vec{B}}{\partial t}=0,$ $\vec{\nabla}\times{}_h\vec{H}=\frac{1}{c}{}_h\vec{J}+\frac{1}{c}\frac{\partial_h\vec{D}}{\partial t},\quad {}_h\vec{D}={}_h\vec{E}+{}_h\vec{P},$ ${}_h\vec{H}={}_h\vec{B}-{}_h\vec{M}_I,\quad \vec{F}={}_hQ\,{}_h\vec{E}+\frac{{}_hQ}{c}(\vec{v}\times{}_h\vec{B})$
esu units	$\vec{\nabla}\cdot\vec{D}=4\pi\rho_Q,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,$ $\vec{\nabla}\times\vec{H}=4\pi\vec{J}+\frac{\partial\vec{D}}{\partial t},\quad \vec{D}=\vec{E}+4\pi\vec{P},$ $\vec{H}=c^2\vec{B}-4\pi\vec{M}_I,\quad \vec{F}=Q\vec{E}+Q(\vec{v}\times\vec{B})$
emu units	$\vec{\nabla}\cdot\vec{D}=4\pi\rho_Q,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,$ $\vec{\nabla}\times\vec{H}=4\pi\vec{J}+\frac{\partial\vec{D}}{\partial t},\quad \vec{D}=\frac{1}{c^2}\vec{E}+4\pi\vec{P},$ $\vec{H}=\vec{B}-4\pi\vec{M}_I,\quad \vec{F}=Q\vec{E}+Q(\vec{v}\times\vec{B})$

To show how the free parameter method, with or without prefixes, works, we apply the first row of Table 4.2 to Eq. (4.1e), reducing it to the rationalized mks system:

$$\vec{D}=\left(\frac{10^{11}}{4\pi c_{\text{cgs}}^2}\frac{\text{farad}}{\text{m}}\right)\vec{E}+\vec{P}. \tag{4.3a}$$

Table 4.2 Free-parameter values for the rationalized mks system which is also called SI units, the unrationalized mks system, Gaussian cgs units, the Heaviside-Lorentz cgs system, esu units, and emu units.

	Rationalization free parameter	Permittivity free parameter	Permeability free parameter	Light-Speed free parameter	Is this a rationalized system?
rationalized mks system, also called SI units	$\Pi = 1$	$\widetilde{\varepsilon} = \dfrac{10^{11}}{4\pi c_{\text{cgs}}^2} \dfrac{\text{farad}}{\text{m}}$	$\widetilde{\mu} = 4\pi \cdot 10^{-7} \dfrac{\text{henry}}{\text{m}}$	$k_0 = 1$	Yes, use $\widetilde{\varepsilon} = {}_f\varepsilon_0$, $\tilde{\mu} = {}_f\mu_0$, and consult Table 4.3(a) to see where the other prefixes go.
unrationalized mks system	$\Pi = 4\pi$	$\widetilde{\varepsilon} = \dfrac{10^{11}}{c_{\text{cgs}}^2} \dfrac{\text{farad}}{\text{m}}$	$\widetilde{\mu} = 10^{-7} \dfrac{\text{henry}}{\text{m}}$	$k_0 = 1$	No, use $\widetilde{\varepsilon} = \varepsilon_0$, $\tilde{\mu} = \mu_0$, and there are no prefixes.
Gaussian cgs units	$\Pi = 4\pi$	$\widetilde{\varepsilon} = 1$	$\widetilde{\mu} = 1$	$k_0 = \dfrac{1}{c}$	No
Heaviside-Lorentz cgs system	$\Pi = 1$	$\widetilde{\varepsilon} = 1$	$\widetilde{\mu} = 1$	$k_0 = \dfrac{1}{c}$	Yes, consult Table 4.3b to see where the prefixes go.
esu units	$\Pi = 4\pi$	$\widetilde{\varepsilon} = 1$	$\tilde{\mu} = \dfrac{1}{c^2}$	$k_0 = 1$	No
emu units	$\Pi = 4\pi$	$\widetilde{\varepsilon} = \dfrac{1}{c^2}$	$\widetilde{\mu} = 1$	$k_0 = 1$	No

Table 4.3(a) Rationalized and unrationalized physical quantities in the rationalized mks system, which is also referred to as SI units.

magnetic vector potential ***Unrationalized***, A	volume current density ***Unrationalized***, J	permeance ***Rationalized***, ${}_f\mathcal{P}$
magnetic induction ***Unrationalized***, B	surface current density ***Unrationalized***, $\mathcal{J}_S$	charge ***Unrationalized***, Q
capacitance ***Unrationalized***, C	inductance ***Unrationalized***, L	resistance ***Unrationalized***, R
electric displacement ***Rationalized***, ${}_fD$	permanent-magnet dipole moment reluctance ***Rationalized***, ${}_fm_H$	reluctance ***Rationalized***, ${}_f\mathcal{R}$
electric field ***Unrationalized***, E	current-loop magnetic dipole moment ***Unrationalized***, m_I	volume charge density ***Unrationalized***, ρ_Q
dielectric constant ***Rationalized***, ${}_f\varepsilon$	permanent-magnet dipole density ***Rationalized***, ${}_fM_H$	resistivity ***Unrationalized***, ρ_R
relative dielectric constant ***Unrationalized***, ε_r	current-loop magnetic dipole density ***Unrationalized***, M_I	elastance ***Unrationalized***, S
permittivity of free space ***Rationalized***, ${}_f\varepsilon_0$	magnetic permeability ***Rationalized***, ${}_f\mu$	surface charge density ***Unrationalized***, S_Q
magnetomotive force ***Rationalized***, ${}_f\mathcal{F}$	relative magnetic permeability ***Unrationalized***, μ_r	conductivity ***Unrationalized***, σ
magnetic flux ***Unrationalized***, Φ_B	magnetic permeability of free space ***Rationalized***, ${}_f\mu_0$	electric potential ***Unrationalized***, V
conductance ***Unrationalized***, G	magnetic pole strength ***Rationalized***, ${}_fp_H$	magnetic scalar potential ***Rationalized***, ${}_f\Omega_H$
magnetic field ***Rationalized***, ${}_fH$	electric dipole moment ***Unrationalized***, p	
current ***Unrationalized***, I	electric dipole density ***Unrationalized***, P	

From the fifth entry of row one, we note that

$$\tilde{\varepsilon} \to \frac{10^{11}}{4\pi c_{\text{cgs}}^2}\frac{\text{farad}}{\text{m}} = {}_f\varepsilon_0;$$

and from Table 4.3(a) we see that

$$\vec{D} \to {}_f\vec{D},$$