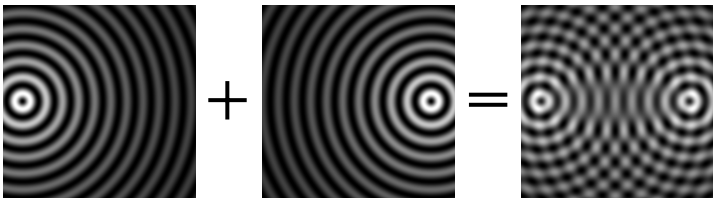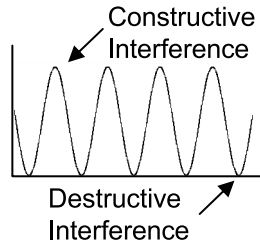## Two-Beam Interference Equation

Interferometric optical testing is based on the phenomena of interference. **Two-beam interference** is the superposition of two waves, such as the disturbance of the surface of a pond by a small rock encountering a similar pattern from a second rock. When two wave crests reach the same point simultaneously, the wave height is the sum of the two individual waves. Conversely, a wave trough and a wave crest reaching a point simultaneously will cancel each other out. Water, sound, and light waves all exhibit interference, but for the purpose of optical testing, the focus will be the interference of light and its applications.

A light wave can be described by its **frequency**, **amplitude**, and **phase**, and the resulting interference pattern between two waves depends on these properties, among others. The **two-beam interference equation** for monochromatic waves is:

$$I(x,y) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2)$$

$$I(x,y) = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi_1 - \phi_2)$$

- I is the **irradiance**. Detectors respond to irradiance, which is the electric field **amplitude**, A, squared:
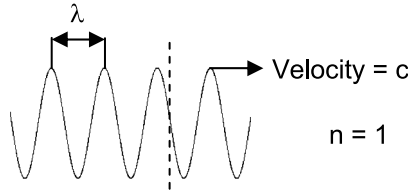
$$I = A^2$$

- ϕ is the **phase** of the wave in radians:

$$0 \leq \phi \leq 2\pi$$

- $\phi_1 - \phi_2 = \Delta\phi$ is the phase difference between the test and reference beams

Basic Concepts and Definitions

- λ is the wavelength of light. Visible light extends from 400 to 700 nanometers

Velocity = c

n = 1

Wave Crests / Second = υ
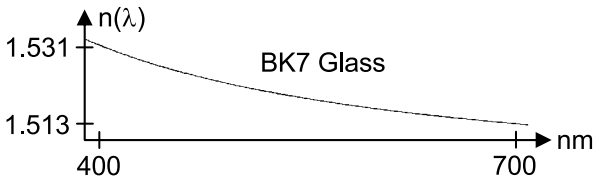
- c is the speed of light in a vacuum ($n = 1$):

$$c = 2.99792 \times 10^8 \frac{\text{m}}{\text{s}}$$

- υ is the optical frequency of the light:

$$\upsilon = \frac{c}{\lambda}; \quad \upsilon = 5.45 \times 10^{14} \text{ Hz for } \lambda = 550 \text{ nm}$$

- n is the **index of refraction** of the medium. n is a function of λ:

$$n(\lambda) = \frac{c}{\text{v}} = \frac{\text{Speed of Light in Vacuum}}{\text{Speed of Light in Medium}}$$

- OPL is the **optical path length** and is proportional to the time light takes to travel from a to b:

$$OPL = \int_a^b n(s)ds; \quad OPL = nt$$

- t is the physical thickness of the medium
- OPD is the **optical path difference** between two beams:

$$OPD = OPL_1 - OPL_2; \quad (\phi_1 - \phi_2) = \frac{2\pi}{\lambda}OPD$$

Conditions for Obtaining Fringes

In order for **interference fringes** to be observed between two beams, several conditions must be met. The light in one beam must be both **temporally** and **spatially coherent** with the other beam in the region where interference fringes are to be observed. In addition, the **polarization** properties of the two beams must be compatible. Finally, the relative irradiances of the two beams must be close in magnitude.

Temporal coherence is inversely proportional to the **spectral bandwidth** of the light source of the two beams. Temporal coherence goes as the **Fourier transform** of the spectral distribution of the source. For example, a **laser** is often modeled as a purely monochromatic source; that is, its spectral bandwidth is zero. The Fourier transform of a zero bandwidth source is a constant, so the temporal coherence of a purely monochromatic source is infinite. Infinite temporal coherence means that light in the one beam can be delayed relative to the second by any amount of OPL (time) and the two beams will still interfere. The temporal coherence of a source is usually given by the **coherence length** (L_c) or the **coherence time** (t_c):

$$L_c = \frac{\lambda_c^2}{\Delta\lambda} \qquad t_c = \frac{L_c}{c}$$

| Source	λ_c (nm)	$\Delta\lambda$	L_c
HeNe Laser	632.8	<0.04 pm	>10 m
Hg Lamp	546	~0.1 nm	~3 mm
SLD	680	12 nm	38 μm
LED	660	25 nm	17 μm
Light Bulb	550	~300 nm	~1 μm

The center wavelength is λ_c and $\Delta\lambda$ is the spectral bandwidth, measured at the FWHM. A source can produce fringes as long as the OPD between the two beams is less than the coherence length. If the magnitude of the OPD between the two beams is greater than the coherence length, fringes will not be observed. As the OPD goes to zero, fringe **visibility** reaches a maximum.

Visibility

Visibility ranges from 0 to 1, where fringes with $V > 0.2$ are usually discernible. The two-beam interference equation should be modified by a temporal visibility function, which depends on the source used and is a function of the OPD. For a laser, $V(OPD)$ goes to 1 and the original two-beam interference equation remains:

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}; \quad I = I_1 + I_2 + 2 \cdot V(OPD) \cdot \sqrt{I_1 I_2} \cos(\Delta\phi)$$

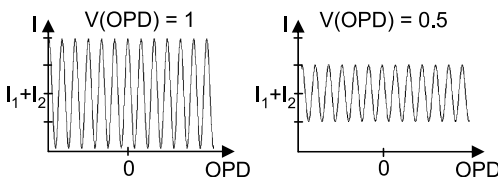Fringe visibility variations due to temporal coherence
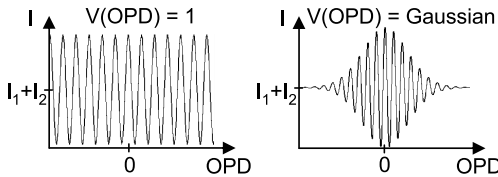
Fringe visibility degradation due to temporal coherence can be improved by decreasing the OPD between the two beams or by spectrally filtering the source.

Spatial Coherence

A second requirement for the observation of interference fringes is that the two beams are spatially coherent. If the source is truly a point source, the two beams are identical except for the OPD. The model of an ideal monochromatic point source gives two beams in which any location in the first beam will interfere with any location in the second beam. Point sources do not exist and real beams have less than ideal spatial coherence. The visibility of the interference fringes for two beams is inversely proportional to the spatial extent of the light source.

Fringe visibility variations due to spatial coherence

Spatial Coherence

Unlike fringe visibility for temporal coherence, fringe visibility degradation due to spatial coherence effects is not a function of OPD. This degradation can be represented in the two-beam interference equation by a constant V_{sc} ($0 \leq V_{sc} \leq 1$) multiplied by the cosine term:

$$I = I_1 + I_2 + 2 \cdot V_{sc} \cdot \sqrt{I_1 I_2} \cos(\Delta\phi)$$

The spatial coherence of a source can be improved by spatially filtering the light source. One way to spatially filter a light source is to use a lens to couple the light into a single mode fiber. Light emerging from the other end of the fiber will fill the numerical aperture of the fiber but will be originating from the central core of the fiber, which is a good approximation of a point source. The core diameter of single-mode fiber is around 5 microns.
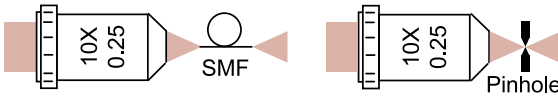
Another approach to spatially filtering a light source is to focus the light onto a small pinhole using a microscope objective. As the magnification of the objective increases, the focused spot size decreases (as does the pinhole size) according to the following equation:

$$D = \frac{1.22\lambda}{NA} \qquad NA = n \sin(\theta)$$
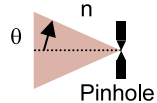
NA is the **numerical aperture** of the objective, where θ is the half angle of the converging beam before the pinhole. The objective creates an 'image' of the source, with the undesirable spatial extent of the source focused off axis. The pinhole blocks all but the central region, and the emerging beam has the spatial coherence properties of a source the size of the pinhole. The chart is for a 1 mm diameter beam at $\lambda = 632.8$ nm.

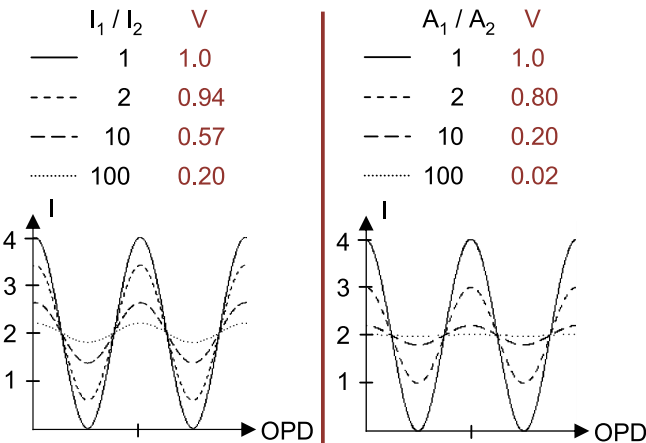| Objective Magnification	NA	Pinhole Diameter
5X	0.10	50 μm
10X	0.25	25 μm
20X	0.40	15 μm
40X	0.65	10 μm
60X	0.85	5 μm

Polarization

Incoherent light, which does not interfere, adds in irradiance, while coherent light, which does interfere, adds in amplitude. In order for two beams to interfere, the electric fields cannot be completely orthogonal. For example, if the first beam is linearly polarized in x and the second is linearly polarized in y, they will not interfere, regardless of the coherence between the two beams. Two beams with orthogonal polarizations do not interfere, while two beams in the same polarization state have maximum fringe visibility. For polarization states in between, fringe visibility depends
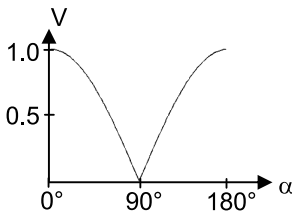
on the dot product of the electric field vectors of the two beams. Fringe visibility between two linearly polarized beams goes as $|\cos(\alpha)|$, where α is the angle between the two states of polarization.

Relative Beam Intensities

Maximum visibility fringes occur when the irradiance of the two beams are equal, which is evident from the two-beam interference equation. As the ratio of the beam intensities deviates from unity, fringe visibility decreases until they are no longer easily detected ($V < 0.2$).

I_1 / I_2	V
—— 1	1.0
- - - - 2	0.94
— - — 10	0.57
·········· 100	0.20

A_1 / A_2	V
—— 1	1.0
- - - - 2	0.80
— - — 10	0.20
·········· 100	0.02

Beamsplitters

Conditions for seeing fringes have been discussed, while methods for creating and recombining two beams have not. Light from one source is split into two beams by a **beamsplitter**, which either divides the wavefront or the amplitude. **Division of wavefront** beamsplitters create two beams from different portions of the original wavefront. A certain degree of spatial coherence is required to see fringes once the beams are recombined since the beams originates from different parts of the wavefront. A classic example of wavefront division is Young's double-slit.

Division of Wavefront

Division of Amplitude
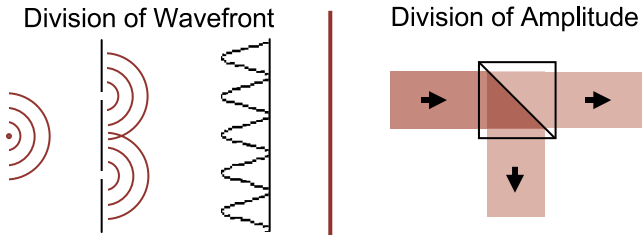
Most laboratory interferometers create two beams using **division of amplitude** beamsplitters, where the irradiance is divided across the entire wavefront such that the diameter of the beam is unchanged. This can be done using cube beamsplitters, plate beamsplitters, pellicles, and diffraction gratings. Light is split as a fraction of incident irradiance in a normal beamsplitter. A **polarization beam splitter**, or **PBS**, splits the light according to its state of polarization, transmitting p-polarized and reflecting s-polarized light. A PBS is one type of **cube beamsplitter**, which in general are made from two right-angle prisms with a coating at the junction of the prisms that gives the desired amount of reflected light. The coating makes it either a PBS or a normal beamsplitter. The external faces of the cube are typically anti-reflection (AR) coated to prevent light loss and spurious fringes. Since light is usually normally incident on the cube face, polychromatic light is not dispersed in a cube beamsplitter.
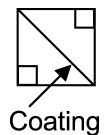
PBS

p

s

Coating

Plate and Pellicle Beamsplitters

Plate beamsplitters are similar to cube beamsplitters in that they divide the amplitude of the incident light and can be made to split the light by polarization or by any desired ratio. One surface is usually AR coated, while the other has the coating to split the beam. Plate beamsplitters can be used to split beams at angles other than 90°. If the source is polychromatic, the index variation in the glass as a function of wavelength will cause a slight spatial displacement between beams of different wavelengths. This is not a concern for sources with a narrow spectrum, such as a laser.

$$d = t \sin(\theta)\left[1 - \sqrt{\frac{1 - \sin^2(\theta)}{n^2 - \sin^2(\theta)}}\right] \approx t\theta\left(\frac{n-1}{n}\right); \quad \theta \text{ in radians}$$
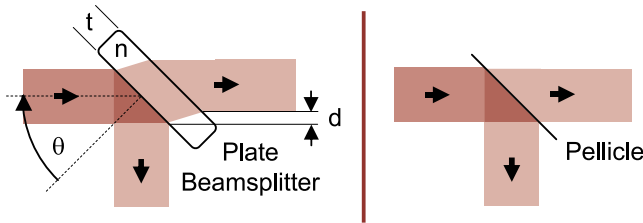

Pellicle beamsplitters are thin ($\sim$5 μm) polymers used like a plate beamsplitter and are most useful for beam sampling. They are not susceptible to wavelength displacement in the beam because they are so thin, but they are fragile and very sensitive to vibrations, making them difficult to use in an interferometer.

Diffraction Grating as a Beamsplitter

A diffraction grating can be used as a beamsplitter for a source that is nearly monochromatic. When placed in a beam, part of the beam continues along the original path, while the diffracted beam leaves the grating at an angle θ_d. The grating period is Λ and m is the diffraction order.

$$\sin(\theta_d) - \sin(\theta_i) = \frac{m\lambda}{\Lambda}$$
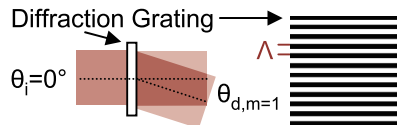
The Interferometer

An interferometer is an instrument that uses the interference of light to make precise measurements of surfaces, thicknesses, surface roughness, optical power, material homogeneity, distances; the list goes on. The resolution of an interferometer is governed by the wavelength of light used and is on the order of a few nanometers. In order to determine the properties of the sample under test, an interferogram is captured and analyzed according to the type of interferometer that created it. Two-beam interferometers return relative information about the OPD between the two beams. Absolute measurements can be made, but extra care must be taken in calibration and system characterization.

Classic Fizeau Interferometer

The classic Fizeau interferometer often uses a spectral emission line to achieve a coherence length of a fraction of a millimeter. The part to be tested is placed on top of a reference flat which has been previously characterized. The **Fresnel reflection** from the bottom of the test piece is the test beam. The reflection from the top surface of the reference flat is the reference beam. The fraction of reflected light ρ at normal incidence is given as a function of the refractive index, n.
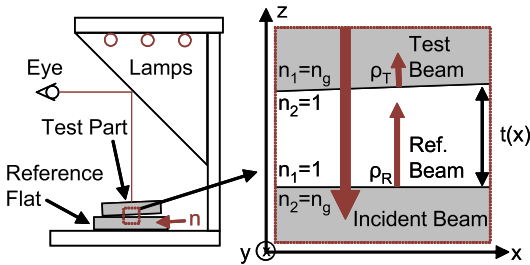

$$\rho = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2 \quad OPD = 2n_{air}t(x) \approx 2t(x); \quad n_{air} \approx 1.0$$

The OPD between the test and reference beams is given above. The reference beam travels the distance between the two surfaces twice; hence the factor of two.

Classic Fizeau Interferograms

The resulting interferogram seen by the eye is shown here for a test part that is tilted in x with respect to the reference part. Any fringe in an interferogram represents a constant OPD. Fringes can be thought of as contour lines on a contour map. Fringes cannot cross each other. The OPD between adjacent bright fringes is one center wavelength of the light source used.
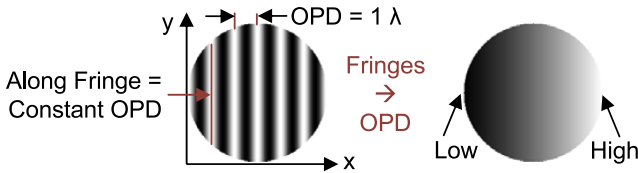
If the test surface has a defect, such as a bump or hole, the fringes will curve around it. To determine if the defect is a bump or a hole on the surface, the direction of increasing OPD must be identified. This can be determined by pushing on one side of the test piece and watching the number of fringes. If the number of fringes increases when pressure is applied, you are pushing on the side with lower OPD. If fringes on the right are higher OPD, meaning the wedge between the two surfaces opens to the right, then this interferogram represents a hole on the bottom surface of the test piece. In the region of the defect, a fringe representing higher OPD is 'pulled in' from the right, indicating that the reference beam traveled further in the region of the defect. Therefore, the defect is a hole with height h, fringe spacing S, and fringe displacement Δ:

$$h = \frac{\lambda}{2}\frac{\Delta}{S}$$
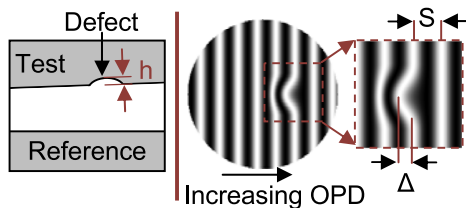
This interferogram is shown with several waves of tilt. Tilt is often purposely introduced so the magnitude of a surface error can be determined. No tilt makes S hard to determine; too much tilt makes Δ hard to find.