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Abstract. The analytical expression for hollow sinh-Gaussian (HsG) beams propagating through a paraxial *ABCD* optical system is derived and used to investigate its propagation properties in a fractional Fourier transform (FrFT) optical system. Several influence parameters of both the HsG beams and the FrFT optical system are discussed in detail. Results show that the FrFT optical system provides a convenient way for modulating HsG beams: HsG beams maintain their dark-centered distribution when the fractional order p is low, and low-ordered HsG beams lose their original dark-centered distribution more quickly than high-ordered ones when the value of p increases. Eventually all HsG beams' intensities evolve into peak-centered distributions with some side lobes located sideways. Furthermore, our results also show that HsG beam intensity distribution versus the fractional order is periodical and the period is 2. The results obtained in this work are valuable for HsG beam shaping. [DOI: 10.1117/1.OE.53.8.086112]

Keywords: hollow sinh-Gaussian beam; fractional Fourier transform; laser beam shaping; propagation.

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1 Introduction

Optical beams with zero central intensity, which are called dark-hollow beams (DHBs), have recently attracted a lot of attention both experimentally and theoretically, due to their unique properties and useful applications in atomic optics, optical communication, optical trapping, and other fields.[1–4] Meanwhile, a rich variety of methods have been used to generate various DHBs, such as the transverse mode selection method,[5] the geometrical optical method,[6] optical holographic method,[7] the computer-generated hologram method,[8] and the hollow optical fibers method.[9] In theory, several models have been proposed to describe DHBs, such as the best-known TEM_{01}^* beam (also known as a doughnut beam), high-order Bessel beams, hollow Gaussian beams,[10–12] etc. In 2012, Sun et al. introduced a new mathematical model called hollow sinh-Gaussian (HsG) beams to depict DHBs.[13] Their propagation characteristics in free space were also studied.

On the other hand, fractional Fourier transform (FrFT) as the generalization of a conventional Fourier transform, was first proposed as a new mathematical tool for solving physics problems by Namias in 1980.[14] Its subsequent potential applications in optics were first explored in 1993 by Mendlovic, Ozaktas, and Lohmann.[15–17] Since then FrFT has become an active research subject in optics and has found wide application in signal processing, optical image encryption, beam shaping, and beam analysis.[18–23] Recently, much work has been done concerning their FrFT for various types of beams that frequently used in modern optics.[24–34] However, to the best of our knowledge, no results have been reported until now about the propagation properties of the newly proposed HsG beams in the FrFT optical system.

In this work, we derived the analytical expression for HsG beams propagating through a paraxial *ABCD* optical system and used it to investigate its propagation properties in the FrFT optical system. The paper is structured as follows, In Sec. 2, a propagation analytical expression for HsG beams through a paraxial optical *ABCD* system is derived. In Sec. 3, evolution of HsG beams' intensity distributions in the FrFT system and their dependent influences on several parameters are discussed in detail and numerically illustrated by using the derived equations. Finally, the mail results obtained are summarized in Sec. 4.

2 Fractional Fourier Transform of Hollow Sinh-Gaussian Beams

From an optical point of view, three kinds of optical systems for performing FrFT are proposed[15–17] and are shown in Fig. 1, which are the Lohmann I system, the Lohmann II system, and the quadratic graded index (GRIN) medium system, respectively. Here, f_s is the standard focal length, $\Phi = p\pi/2$ with p being the fractional order, and z is the axial distance between the input and output planes along the optical axis in the GRIN medium. According to matrix optics, the transfer matrix for Lohmann I optical system can be expressed as

$$R_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} 1 & f_s \tan(\Phi/2) \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ -\sin\Phi/f_s & 1 \end{pmatrix} \begin{bmatrix} 1 & f_s \tan(\Phi/2) \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} \cos\Phi & f_s \sin\Phi \\ -\frac{\sin\Phi}{f_s} & \cos\Phi \end{pmatrix}. \quad (1)$$

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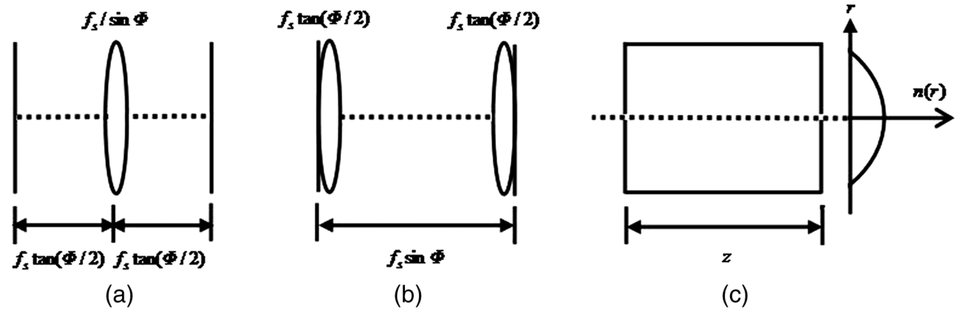

Fig. 1 Three kinds of optical systems for performing fractional Fourier transform (FrFT): (a) the Lohmann I system, (b) the Lohmann II system, and (c) the quadratic graded index (GRIN) system.

For a Lohmann II optical system, the corresponding transfer matrix can be described by

$$R_2=\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{bmatrix}1 & 0\\ -\tan(\Phi/2)/f_s & 1\end{bmatrix}\begin{pmatrix}1 & f_s\sin\Phi\\ 0 & 1\end{pmatrix}\begin{bmatrix}1 & 0\\ -\tan(\Phi/2)/f_s & 1\end{bmatrix}=\begin{pmatrix}\cos\Phi & f_s\sin\Phi\\ -\frac{\sin\Phi}{f_s} & \cos\Phi\end{pmatrix}. \tag{2}$$

For the GRIN medium system, the transfer matrix with quadratic index variation $n(r)=n_0[1-r^2/(2a^2)]$, can be written as[35]

$$R_3=\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{bmatrix}\cos(z/a) & a\,\sin(z/a)\\ -\frac{1}{a}\,\sin(z/a) & \cos(z/a)\end{bmatrix}. \tag{3}$$

In Eq. (3), a denotes the radius of the GRIN medium. Obviously, Eqs. (1), (2), and (3) have the same form when $f_s=a$ and $\Phi=z/a$. Hence, the above-mentioned three optical systems have the same transfer matrix and they are equivalent for performing FrFT.

In a cylindrical coordinate system, the electric field of the HsG beams in the original plane ($z=0$) is defined by[13]

$$E_n(r,0)=G_0\sinh^n\left(\frac{r}{w_0}\right)\exp\left(-\frac{r^2}{w_0^2}\right). \tag{4}$$

In Eq. (4), n ($n=0,1,2,\cdots$) denotes the order of the HsG beams and G_0 is a constant related to the beam power. Obviously, for $n=0$, the beam governed by Eq. (4) is the conventional fundamental Gaussian beam with a beam waist of w_0. However, for $n\geq 1$, a new kind of HsG beam is obtained.

As defined by Eq. (4), the amplitude of the HsG beam is determined by the beam's order n and waist size w_0. In order to visualize the shape of HsG beams, a preliminary demonstration is shown in Fig. 2 for HsG beams with different orders n [Fig. 2(a)] and with different waist sizes w_0 [Fig. 2(b)], respectively. All curves in Fig. 2 have been normalized to their peak intensity value. It is apparent from Fig. 2 that the irradiance profile of the HsG beam presents a single bright ring, and the central dark size increases as n and w_0 increase. Therefore, one can control the intensity distribution of HsG beams by choosing n and w_0.

On the other hand, Eq. (4) can be rewritten in the form

$$E_n(r,0)=G_0\sum_{m=0}^{n}a_m b_m\,\exp\left[\frac{-(r+c_m)^2}{w_0^2}\right], \tag{5}$$

with the coefficients given by

$$a_m=(-1)^m 2^{-n}\binom{n}{m},$$
$$b_m=\exp\left[\left(m-\frac{n}{2}\right)^2\right],$$
$$c_m=w_0\left(m-\frac{n}{2}\right), \tag{6}$$

where $\binom{n}{m}$ is a binomial coefficient. Equation (5) indicates that the n'th order HsG beam can be generated in a laboratory by the superposition of several decentered Gaussian beams with the same waist width w_0, whose centers are located at positions $(-c_m,0)$, respectively.

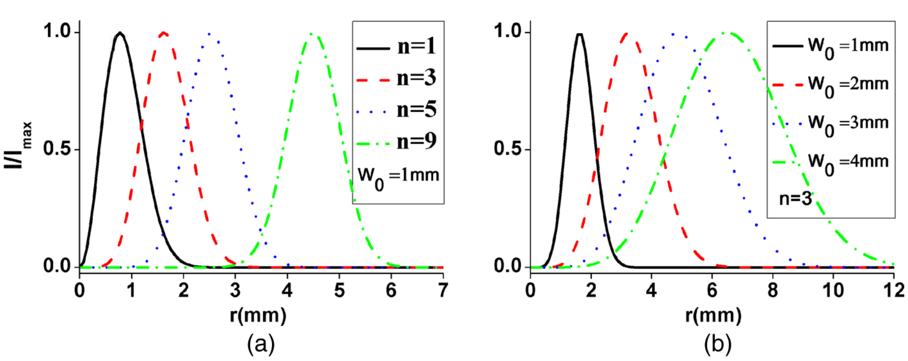

Fig. 2 Normalized intensity distribution of hollow sinh-Gaussian (HsG) beams as a function of r, (a) for different orders n and (b) for different waist sizes w_0.

Within the framework of the paraxial approximation, the propagation of any laser beam through an *ABCD* optical system can be described by the generalized Huygens-Fresnel diffraction integral, known as Collins integral formula, which takes the following form in a cylindrical coordinate system:[36,37]

$$E_n(r,z)=\frac{i}{\lambda B}\exp(-ikz)\times\int_0^{2\pi}\int_0^{\infty}E_n(r',0)$$
$$\times\exp\left\{-\frac{ik}{2B}[Ar'^2-2rr'\cos(\theta-\theta')+Dr^2]\right\}r'\mathrm{d}r'\mathrm{d}\theta'. \quad (7)$$

In Eq. (7), $E_n(r',0)$ and $E_n(r,z)$ are the electric fields in the input and output planes, respectively. r', θ' and r, θ are the radial and azimuthal angle coordinates in the input and output planes, respectively. z is the axial distance between the input and output planes along the optical axis. k is the wave number related to the wavelength λ by $k=2\pi/\lambda$. A, B, C, and D are the transfer matrix elements of the optical system between the input and output planes, respectively.

Substituting Eqs. (5) and (6) into Eq. (7), and recalling the following integral equation:

$$J_0(t)=\frac{1}{2\pi}\int_0^{2\pi}\exp(it\cos\varphi)\mathrm{d}\varphi, \quad (8)$$

we can transform Eq. (7) as

$$E_n(r,z)=\frac{iG_0k}{B}\exp(-ikz)\exp\left(-\frac{ik}{2B}\cdot Dr^2\right)\sum_{m=0}^{n}a_mb_m$$
$$\times\int_0^{\infty}\exp\left(-\frac{ikA}{2B}r'^2\right)\cdot\exp\left[\frac{-(r'+c_m)^2}{w_0^2}\right]\cdot J_0\left(\frac{kr}{B}r'\right)\cdot r'\mathrm{d}r'. \quad (9)$$

Expand the exponential part into a Taylor series and use the integral equation of the hypergeometric Kummer function to evaluate the beams. When $\mathrm{Re}(\mu+v)>0$ and $\mathrm{Re}(a^2)>0$, the integral equation will take the following form:[38]

$$\int_0^{\infty}t^{\mu}\exp(-a^2t^2)J_v(pt)\mathrm{d}t$$
$$=\Gamma\left(\frac{\mu+v+1}{2}\right)\frac{p^v}{2^{v+1}a^{\mu+v+1}\Gamma(v+1)}$$
$$\times{}_1F_1\left(\frac{\mu+v+1}{2},v+1;-\frac{p^2}{4a^2}\right). \quad (10)$$

In Eq. (10), $J_v(x)$ stands for the v-order Bessel function of the first kind, $\Gamma(x)$ denotes the gamma function, and ${}_1F_1(a,b;x)$ is the confluent hypergeometric function, respectively. After tedious but straightforward integration, we obtain a result as follows:

$$E_n(r,z)=\frac{iG_0k}{2B}\exp(-ikz)\exp\left(-\frac{ikDr^2}{2B}\right)$$
$$\times\sum_{m=0}^{n}\sum_{s=0}^{\infty}(-1)^m2^{-n}\binom{n}{m}\frac{(n-2m)^s}{w_0^s\cdot s!}\times\Gamma\left(1+\frac{s}{2}\right)$$
$$\cdot\left(\frac{1}{w_0^2}+\frac{ikA}{2B}\right)^{-1-s/2}\cdot{}_1F_1\left[1+\frac{s}{2},1;-\frac{\left(\frac{kr}{2B}\right)^2}{\left(\frac{1}{w_0^2}+\frac{ikA}{2B}\right)}\right]. \quad (11)$$

Equation (11) is the general propagation and transformation for HsG beams through a paraxial *ABCD* optical system, which provides a convenient and powerful tool for

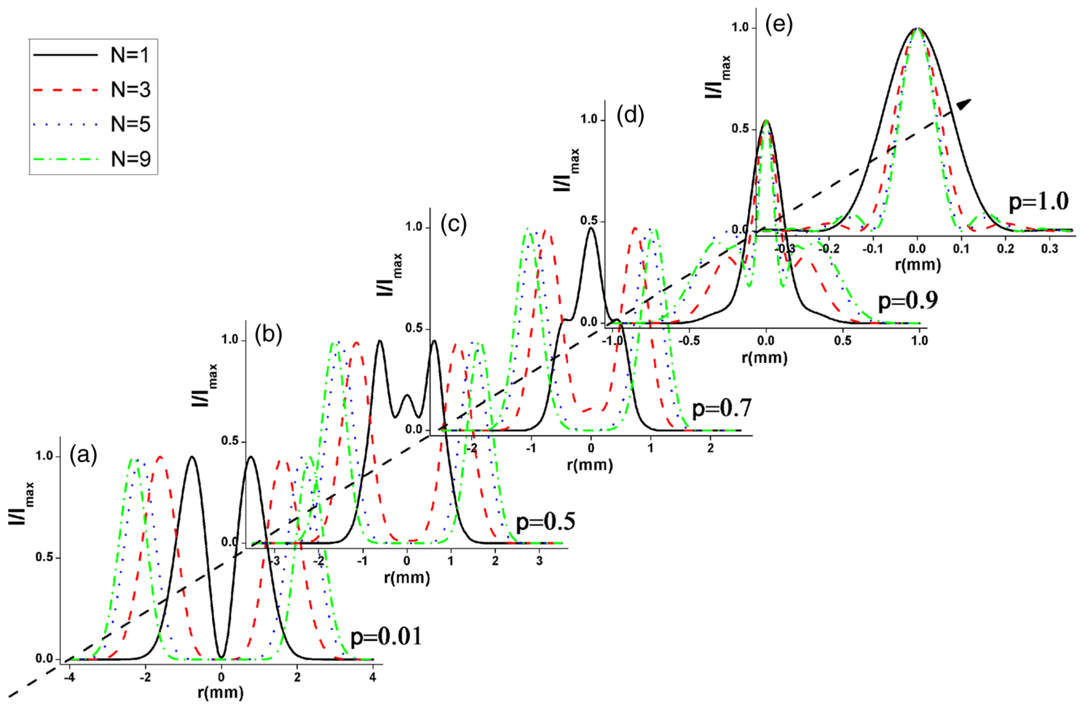

Fig. 3 Normalized intensity distribution of HsG beams in the FrFT planes with different fractional orders p for Lohmann I and Lohmann II optical systems.

treating the propagation and transformation of HsG beams. Substituting Eq. (1) or (3) into Eq. (11), one obtains the analytical expression for the hollow HsG beams propagating through the FrFT optical system. Then the corresponding intensity distribution reads as

$$I_n(r,z) = E_n(r,z)E_n^*(r,z). \tag{12}$$

In the following, we investigate the HsG beam's intensity distribution evolution in the FrFT system.

3 Numerical Simulations and Analysis

According to the analytical expressions obtained in Sec. 2, in the following, we numerically investigate the properties of HsG beams propagating through the FrFT system. In the following discussion, influence factors of HsG beam order n and the fractional transform order p on the evolution of the beam's intensity distribution in the FrFT optical system are considered. Without loss of generality, we choose the calculation parameters of HsG beams as $\lambda = 0.632$ μm, $w_0 = 1$ mm, and $n = 1$, 3, 5, 9.

Figure 3 depicts the normalized intensity distribution of HsG beams in several FrFT planes. For convenience of comparison, we choose the HsG beams of three different orders (i.e., $n = 1$, 3, 5, 9). It can be seen from Figs. 3(a) and 3(b) that HsG beams maintain their dark-centered distribution when the fractional order p is low, and low-ordered HsG beams lose their original dark-centered distribution more quickly than high-ordered ones when the value of p increases [see Fig. 3(c)]. Eventually all beams evolve into a peak-centered distribution with some side lobes located sideways [see Fig. 3(e)]. Furthermore, higher-ordered HsG beams have sharper center-peaked distributions. This indicates that in order to reshape the HsG beams, one can choose reasonable optical parameters for HsG beams in the optical FrFT system.

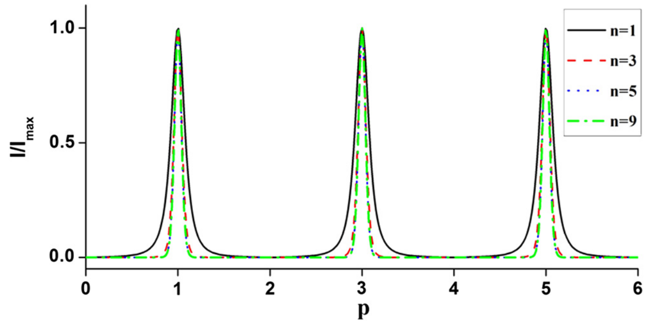

Fig. 5 Evolution of the on-axis intensity of HsG beams of several orders n in the FrFT planes versus the fractional order p.

Figure 4 shows the normalized intensity distribution of HsG beams in different planes in the GRIN medium. In Fig. 4, we choose $a = f_s = 1$ m; hence, the GRIN medium is equivalent with the Lohmann I and Lohmann II optical systems for performing FrFT. Obviously, in our case, the planes $z = 0.079$, 1.38, 1.57, 1.76, and 3.06 m in the GRIN medium are, respectively, equivalent to $p = 0.05$, 0.88, 1.0, 1.12, and 1.95 in the FrFT optical system. Furthermore, one can find from Fig. 4 that the intensity distribution is symmetrical about $z = 1.57$ m (i.e., $p = 1$).

Figure 5 shows the evolution of the normalized on-axis intensity distribution of HsG beams of several orders n in the FrFT planes versus the fractional order p. It is obvious from Fig. 5 that the dependence of the normalized on-axis intensity on the fractional order p is periodic, and the period is 2. The on-axis intensity has a maximum value when $p = 2n + 1$ and a minimum value when $p = 2n$.

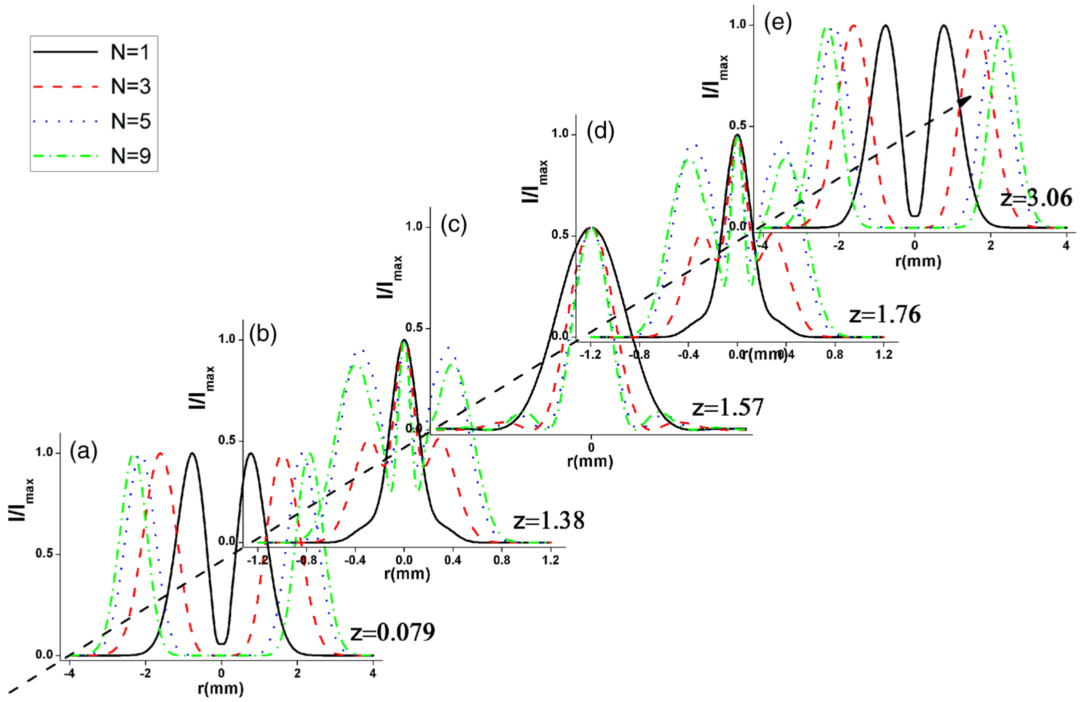

Fig. 4 Normalized intensity distribution of HsG beams in different planes in the GRIN medium.

4 Conclusions

In this work, we have derived the analytical expression for HsG beams propagating through a paraxial *ABCD* optical system and used it to investigate its propagation properties in the FrFT optical system. Several influencing parameters of both the HsG beams and the FrFT optical system are discussed in detail. Results show that the FrFT optical system provides a convenient way for modulating HsG beams: HsG beams maintain their dark-centered distribution when the fractional order p is low, and low-ordered HsG beams lose their original dark-centered distribution more quickly than high-ordered ones when the value of p increases. Eventually, all HsG beams' intensities evolve into peak-centered distributions with some side lobes located sideways. Furthermore, our results also show that HsG beam intensity distribution versus the fractional order is periodical and the period is 2. The results obtained in this work are valuable for the HsG beam shaping.

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