On Teaching Temporal Coherence and the Wiener Khintchin Theorem at a Senior/Graduate Level

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ABSTRACT

One important issue in teaching interferences is that two separate wavelengths usually do not interfere: any interference pattern is the spectral integral of all interference patterns of all monochromatic components. Although optical detectors are quadratic in nature, crossed terms involving two different frequencies in the expression of an interference pattern vanish. More precisely, while in non stationary signals such as ultrashort pulses two wavelengths can give rise to beating phenomena, this does not happen with the usual thermal light beams. The phenomenon is directly connected with the Wiener Khintchin theorem, and therefore with the principle of Fourier transform spectroscopy.

In an introductory course, oversimplification leading to frustrating physical and mathematical deficiencies is hardly avoidable. In this communication, we suggest an introduction of this question at a senior/graduate level. Numerical simulations are used to provide an intuitive understanding of the phenomena. If the Wiener Khintchin theorem is introduced by defining the power spectrum from the infinite time limit of the ordinary Fourier transform of a gaussian windowed version of the signal, the mathematics are simple and the method offers a clear connection with the operation of real (i.e. finite time) detectors analysing interference fringes.

Keywords: coherence, interferences, classical detection noise, education

1. INTRODUCTION

When teaching interferences, the lecturer always encounters the question: why don't two different frequencies ever interfere? A two-beam interferometer illuminated by light of a different frequency on each arm does not show fringes. Or to be more precise, in the quite common case of thermal light sources[1], common experience shows that interferences between two different frequencies of the source spectrum cannot be observed. Interferences with a non strictly monochromatic light are observed but merely result from the incoherent superposition of all component monochromatic interference intensities. Providing a convincing explanation of this fact in an elementary course is not easy because statistical tools are required that are available to the students only at a more advanced level – typically, senior undergraduate or graduate. However, a full account of the phenomenon based on solid statistical grounds is not even commonly found in textbooks designed for such advanced students. Such is the aim of the present chapter.

Indeed, the superposition of two different frequencies should be expected to show a beat phenomenon in time. Nevertheless, common experience shows that interference effects are stationary in time and show no beat effect. If we start from the definition of interferences as the phenomenon whereby the intensity resulting from the superposition of two light beams differs from the sum of the two intensities, we are led to clarifying the temporal aspects in the detection of light beams.

First the common situation of describing the effect at a less advanced level is examined, together with the difficulties that automatically arise. After a review of the necessary statistical concepts such in particular as the Wiener Khintchin theorem, the issue of what exactly is meant by average intensity and the associated "classical detection noise" is addressed, leading to a complete model of temporal coherence in thermal light. Numerical simulations are used to give a more intuitive feeling of the phenomena.

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2. ELEMENTARY APPROACH

This section summarizes a typical description of temporal coherence conditions at an introductory level that may, depending on physics teaching organization, vary from the end of high school to college junior level (see e.g. Ref. 2,3,4).

2.1. Light detection model

Describing interferences implies a model for light detection. We shall adopt the standard model[5] of a quadratic detector that integrates the instantaneous intensity of a scalar field over a time duration T. Specifically, the "light disturbance" is described as a scalar $U(\vec{r},t)$, a complex function of space and time. The standard ("analytical signal") complex notation of real quantities will be used throughout the paper. The instantaneous intensity is defined as

$$i(\vec{r},t)=\left|U(\vec{r},t)\right|^2 \tag{1}$$

and the detector response is a weighted average of the instantaneous intensity commonly known as just "intensity"[6]:

$$I(\vec{r},t)=i(\vec{r},t)*R(t) \tag{2}$$

where the star denotes convolution and R is the detector response function, a real valued function of characteristic width T.

The numerical simulation of Fig. 1 illustrates a short time scan of the instantaneous intensity of a source with gaussian spectrum. In our simulations, we have assigned independent random phases, uniformly distributed between 0 and 2π, to all points in the square root of the simulated power spectrum, then computed the inverse Fourier transform to obtain the light disturbance function U versus time t. Then the real part was taken both for plotting the figures and for computing equation (2), which is a slight difference with the analysis in this manuscript but bears essentially no consequence on our results. The advantage is that it shows the quasi-sinusoidal behavior of light.

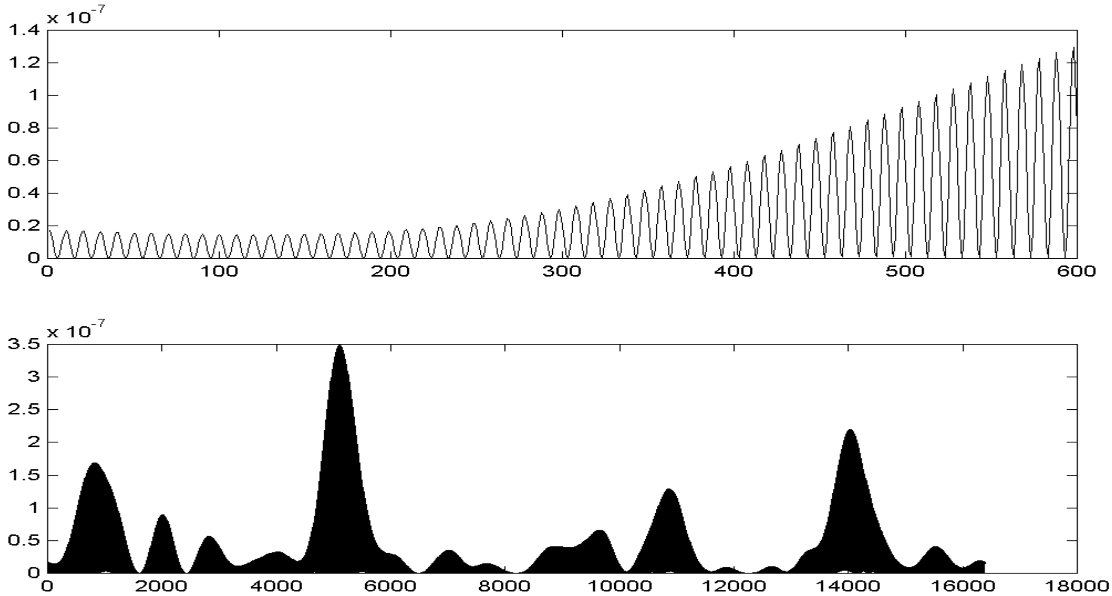


Figure 1. Instantaneous intensity of simulated optical disturbance with gaussian power spectrum. Upper part is an enlarged fraction of lower part. Time scale: 20 time units per average period of disturbance. Intensity scale is arbitrary.

This model relies on the fact that photoelectronic and photochemical detectors, i.e. the vast majority of detectors, are too slow to linearly respond to the optical field but respond quadratically in an electric dipolar mode. We use the so-called "scalar approximation", under which U designates a typical component of the electric part of the light electromagnetic field, which is strictly correct in the case of linear polarization and will be sufficient for the purpose of this chapter although a more complete investigation of the noise effects that will be described below shows that they do depend on the polarization state. The response function R is needed to introduce the essential fact that response time T is very slow compared to the characteristic time scale of the light scalar U: typically, visible light pseudo-periods are on the order of a few femtoseconds while the fastest photoelectric detectors, which are optical telecommunication photodiodes, can barely follow the pace with 40 Gb/s signals, which means that their response time is on the order of a few picoseconds. Indeed, response time T extends over quite a large range, from a few picoseconds to seconds and even more.

To simplify dimensionality considerations, R will be assumed normalized:

$$\int_{\mathbb{R}} R(t)\,dt = 1 \tag{3}$$

Indeed, detectors are causal, therefore R is zero for negative time. Whenever useful for the sake of illustration, a decreasing exponential model will be used for R, which is both convenient analytically and plausible in many cases since it corresponds to an RC time constant limited photoelectric detector:

$$R(t) = \frac{1}{T}\,\mathrm{Hea}(t)\exp\left(\frac{-t}{T}\right) \tag{4}$$

where Hea is the Heaviside step function. In fact, a standard rectangular response could have been used as well, and the main results of this chapter do not rely on any particular shape of response function R. Also, analytically, a gaussian model is even more convenient: it is appropriate for deriving the Wiener Khintchin theorem as mentioned in section 3 below. But it does not well describe the response of any common detector and is not even causal; therefore for numerical simulation of light detectors we shall stick to the more realistic decreasing exponential.

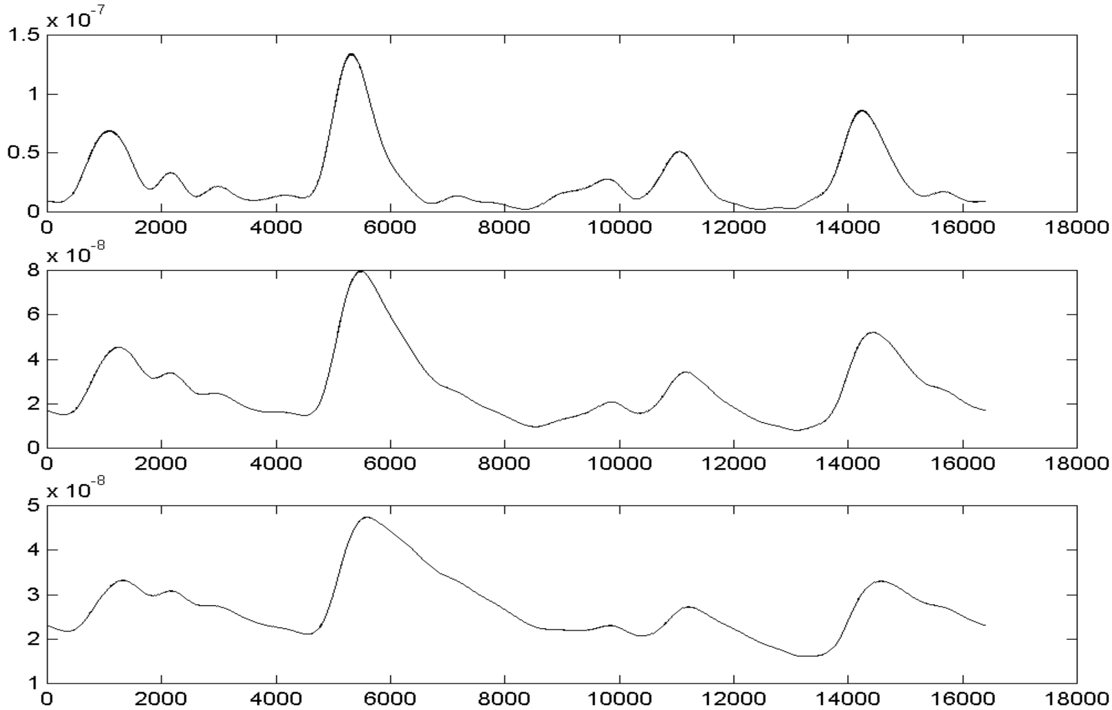


Figure 2. Detected intensity with three imaginary detectors whose response time corresponds, respectively, to 15, 50 and 150 average pseudo-periods of the disturbance.

In Eq. (2), the space coordinate has been omitted from response function R. Indeed, it could be argued that detectors respond to a weighted average over space as well as over time, but space effects are not covered by this article and are also not nearly as rich as the time effects in terms of the physical phenomena involved in the present context. Indeed, space coordinate $\vec{r}$ will be omitted altogether in the following. This assumes that the detector size is much smaller than the characteristic scale of intensity variations; in particular, the detector size is much smaller than any interference fringe.

Fig. 2 shows the behavior of the light disturbance of Fig. 1 as seen through imaginary detectors of three different response times. Realistic response times are in fact much longer. While an ideally fast detector would in principle follow the instantaneous intensity fluctuations, detectors with longer response times tend to blur out fast variations, thereby effectively estimating the time average of instantaneous intensity.

2.2. Observing two beam interferences with two distinct frequencies

Consider a two beam interferometer with its two arms illuminated by two hypothetical strictly monochromatic beams of frequencies ν_1 and ν_2: the detected intensity at the output after the two beams combine on the detector is:

$$I(t) = I_0 \left| \exp(2i\pi\nu_1 t + i\varphi_1) + \exp(2i\pi\nu_2 t + i\varphi_2) \right|^2 * R(t) \tag{5}$$

where I_0 is a constant, φ_1 and φ_2 are two phases that take into account propagation delays over both channels, and it has been assumed for simplicity that the two beams separately produce the same intensity. Denoting by FT the Fourier transform operator and by $\tilde{f}$ the Fourier transform of function f, and using the fact that R is real-valued, i.e.

$$\tilde{R}(\nu) = \left|\tilde{R}(\nu)\right| \exp\left(i \arg \tilde{R}(\nu)\right) = \tilde{R}^*(-\nu) \tag{6}$$

(5) implies after little algebra:

$$\begin{aligned} I(t) &= 2I_0 \left[1 + \cos\left(2\pi(\nu_1 - \nu_2)t + \varphi_1 - \varphi_2\right) * R(t)\right] \\ &= 2I_0 \left[1 + \left|\tilde{R}(\nu_1 - \nu_2)\right| \cos\left(2\pi(\nu_1 - \nu_2)t + \varphi_1 - \varphi_2 + \arg \tilde{R}(\nu_1 - \nu_2)\right)\right] \end{aligned} \tag{7}$$

The first term in both right hand side expressions is the "incoherent" superposition of the two interfering beams, while the term with the cosine expresses the interferences in the form of a beat term. The first line of Eq. (7), however, explicitly shows the beat term between the two interfering frequencies as being filtered by the low pass filtering effect of response function R, which in the second line results in the interference term being damped in amplitude. Therefore, in such a first attempt at describing interferences as the phenomenon whereby two superposed beams produce an intensity that differs from the sum of the two component intensities, the interference term appears to vary in time, which is not supported by experimental evidence. To explain that indeed, such terms in practice do not occur, one can resort to typical order of magnitude calculations. These will be done here under the decreasing exponential model of Eq. 4, for which the damping factor takes on the form:

$$\left|\tilde{R}(\nu_1 - \nu_2)\right| = \frac{1}{\sqrt{1 + 4\pi^2(\nu_1 - \nu_2)^2 T^2}} \tag{8}$$

2.3. Discussion

Assume that the two line sources are, e.g., the two classical sodium D lines, of respective center wavelengths 599,0 nm and 599,6 nm, and that the detector is the eye, T=0,1 s. From Eq. (8), the damping factor is 3.10^{-9}, which obviously means that the beat phenomenon is invisible. The typical conclusion is that two different frequencies cannot interfere and that any interferometer must be illuminated by light of the same spectral content on both arms, or more precisely, in this elementary description, by the same *monochromatic* light on both arms. In that case, the damping factor, according to Eq. (8), is unity, time dependence in Eq. (7) vanishes, and stable interferences can be observed. However, no light is strictly monochromatic, and yet interferences may be observed in appropriate circumstances with sources of any arbitrary spectral width. A more complete analysis is therefore needed.

Consider now that the interferometer is illuminated by two *different* sources that both, after suitable spectral filtering, emit only the D_1 line of sodium. The D_1 line has a narrow spectrum, but of course it is not strictly speaking monochromatic. This situation is usually analyzed in terms of wave packets (or wave trains) being emitted by the two sources and showing no phase correlation; this amounts to saying that phases φ_1 and φ_2 are piecewise constant functions of time. During the period of time when two wave trains arrive together at the detector, φ_1 and φ_2 remain constant and Eq. (8) predicts unity contrast, but the time duration of wave trains is very short, so that during time T this model amounts to adding many contributions of the type of Eq. (7) with randomly distributed values of $\varphi_1 - \varphi_2$. More realistically, phases φ_1 and φ_2 are both fast varying, uncorrelated random functions of time. The quite correct conclusion of this discussion is that *on the average* no interference appears. Now, does this mean that *at any given time* some beat contrast is left or not, perhaps showing fast randomly moving fringes, or does this mean that Eq. (8) is not applicable for some reason that remains to be clarified?

As should be clear from the above, this kind of argument implies a direct relation between the impossibility of obtaining interferences with two different light sources emitting the same spectral line and their lack of monochromaticity, i.e. the width of that line. Then, the issue of the relation between spectral content and interference visibility arises again, and some more sophisticated version of Eq. (8) is required to show that in a typical situation, not only must both interferometer arms be illuminated by light of the same spectral content, but they must even be illuminated by light from one and the same source. The issue, however, is not trivial: one may ask what happens if two ideal, very narrow filters extract two spectral components from one line, say the sodium D_1 line, that are extremely close to each other, say 100 Hz apart or 10 Hz apart, and each one is sent to one arm: the damping factor would not be small compared to unity, and *beat effects should be visible*. Now, this is impractical because on the one hand, no such filters are available, and on the other hand, the amount of energy available from a typical thermal source (we exclude lasers from our discussion) in such extremely narrow spectral regions is too small to give rise to fringes visible by the eye. But even if we forget about these hypothetical filters, these narrow spectral components are part of the light emitted by the source, and there are even infinitely many of them, all adding up their own beat effects in the interference field. Therefore we are back to the same question: the result *averages to* zero indeed, but how can we assert that it *is always* zero?

This kind of fuzzy discussion is unavoidable in an elementary presentation of interferences, and yet it is, at least in our opinion, always fairly confusing to the students and embarrassing to an instructor who wishes to avoid postulating that two different sources with the same spectrum and two different frequencies from one given source cannot interfere. Whence the need of the formally more satisfactory discussion that follows, which unfortunately is accessible to students only at a fairly advanced stage.

The need for this kind of clarification may in fact even be more obvious nowadays with the fast detectors that have been available for a few years only. Let us go back to the D_1-D_2 combination. If the detector responds as fast as an optical telecommunications photodiode, T=10 ps, the damping factor increases from the above-mentioned 3.10^{-9} to 0,9995, and therefore the beat frequency is clearly resolved and the phenomenon can be followed. Even with one red line (700 nm) and one violet line (400 nm), the damping factor with this ultrafast detector does not fall to less than 5% so that something is left over even from two lines spanning the whole visible spectrum. Even the question of what happens with a continuous spectrum makes sense.

As a final comment in this context, one should note that the interferometer structure did not play any role in this discussion: one can just as well think of Eq. (5) as expressing two spectral components from one given source reaching a detector, independently of whether there is an interferometer in-between or not. Therefore, the issue of possible random beat phenomena arising by just illuminating a detector with a source arises: indeed, the answer is “classical detection noise” does exist, and after reviewing the applicable analytical tools we shall first review it before coming back to our core subject, interferences.

Fig. 3. shows the instantaneous intensity created by the two doublet lines. While the beat phenomenon is conspicuous, it is averaged out by any realistic detector.

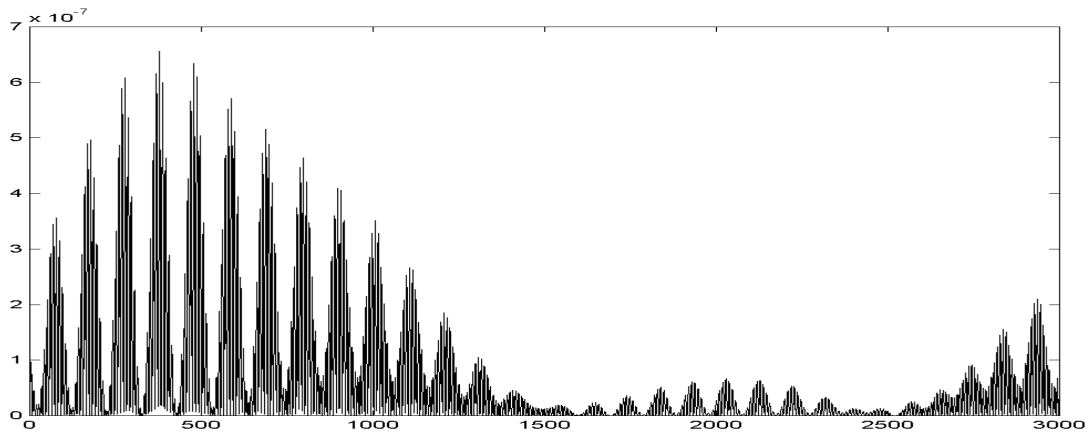


Figure 3. Instantaneous intensity of a doublet.

Fig. 4 relates to the two sources emitting the same spectral line. To visualize the distinction with the single source case, four instantaneous intensity curves are shown. The top two are the two "interfering" disturbances. The third is the sum, while in the last one disturbance has been delayed by half a pseudo-period in time with respect to the other, showing a random beat which is to a good extent complementary to that of the previous curve. Therefore, fast time effects do exist, but all detectors average out all details in the time domain and no interference are visible.

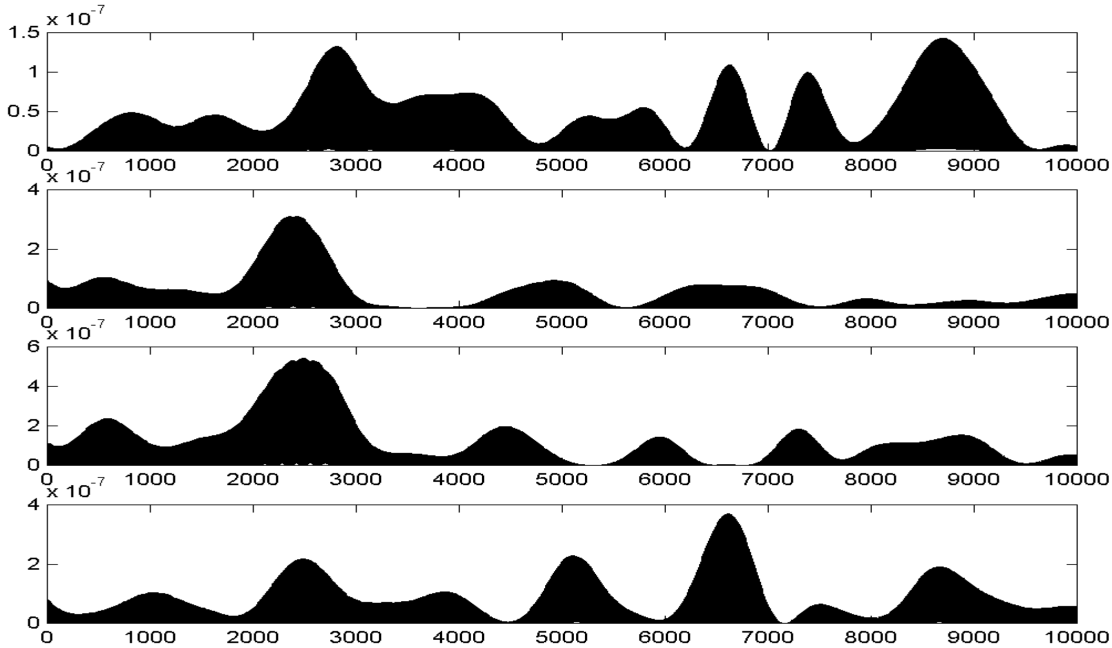


Figure 4. Instantaneous intensities, from top to bottom: first disturbance; second disturbance; sum of the two disturbances; sum with second disturbance shifted half a pseudo-period (10 time units).

3. LIGHT AS A STATIONARY STOCHASTIC FUNCTION AND ITS POWER SPECTRUM

Light emission detail is beyond practical control, except to an extent in laser sources, which are not considered here. Spontaneous emission is a random phenomenon. It therefore makes sense to approach temporal coherence and interferences using stochastic functions. Henceforth, we consider the complex amplitude U(t) as a stochastic process and denote by $U_\mu(t)$ one realization. To proceed, we first need to clearly define the associated power spectrum. This raises the following issues:

- any real physical source starts emitting at one time and stops at some later time, therefore the complex amplitude U(t) that the source sends at some point in space is of bounded support. This is true both for U and for its realization U_μ.
- Being of bounded support, U_μ has a Fourier spectrum, whose modulus square might appear to be an appropriate starting point to define a power spectrum; the definition of spectrum of the stochastic process U, however, is at this stage an open question.
- In addition, finite support effects are not relevant here. On the contrary, everyday life experience as well as the classical use of spectrometers in physics laboratories suggest that one should be able to obtain information on the power spectrum from a small sample in time, and that in common cases this spectrum is stable in time. Explicitly accounting for time support effects that describe the finite duration of source emission is of marginal interest and can only lead our analysis astray. It will therefore simplify the model if time support bounds of the source emission are ignored. This amounts to saying that we consider U(t) as a stationary stochastic process inside a time window which is so broad that its boundaries are of no concern in what follows (Fig. 5): it is just as simple to consider U(t) as stationary altogether. As will appear shortly, stationary covariance is the only required type of stationarity. Simplifying the model by ignoring its time support, unfortunately, does not directly make the mathematics simpler, because in this case also U_μ has no spectrum except in the sense of distributions, and its power spectrum then does not exist[7].

The solution to these questions is the standard definition of the power spectrum of a stochastic process with stationary covariance. This is detailed in a companion paper to this communication[8]. In short

- the power spectrum of stochastic process U, S_U, can conveniently be defined as

$$S_U\left(\nu_o\right) = \lim_{T\to\infty} \mathrm{E}\left(\left| \mathrm{U} \otimes \frac{\exp\left(-\pi \frac{t^2}{2T^2} + 2i\pi\nu_o t\right)}{\sqrt{T}}\right|^2\right) \quad (9)$$

where E stands for the ensemble average (or mathematical expectation) and symbol $\otimes$ indicates correlation, and

- the Wiener-Khintchin theorem states that the statistical autocovariance of U and its power spectrum form a Fourier transform pair:

$$\mathrm{Cov}_{UU} \xrightarrow{TF} S_U \quad (10)$$

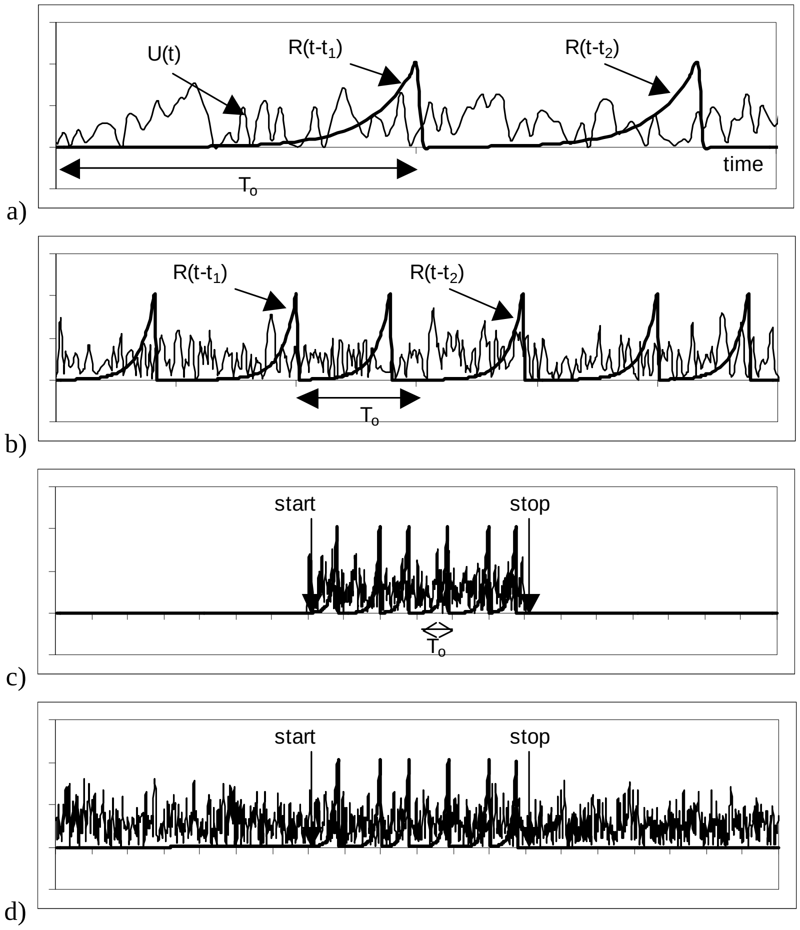


Figure 5. a) Light detection at two different times t_1 and t_2 averages in time over sections of the random light disturbance function U that are not identical. Averaging is suggested by the exponential response functions R. b) During long periods of time, in usual situations, light sources are macroscopically stationary: observations at various times t_1, t_2,… give identical results. c) However, sources are not really stationary but start and stop emitting at some given instants in time. d) Nevertheless, for experiments performed during the emission of light, it does not make any difference if the light disturbance function U is extrapolated to be stationary, which simplifies the model. The time scale of the simulations is suggested by the arbitrary time interval T_o.

Immediate consequences of these relations include the fact that the complex amplitude covariance width, which is a good measure of the coherence time of the source, is inversely related to the source bandwidth.

From these results, we now proceed with the investigation of the fluctuations that may be expected when detecting the intensity. We first consider a detector that directly receives $U_\mu(t)$, and then a detector behind a two beam interferometer illuminated by complex amplitude $U_\mu(t)$. While the Wiener-Khintchin theorem applies to the ensemble average, what the detector senses is not directly an ensemble average: our goal is not only to express the relation between the intensity and its ensemble average, but also to examine the corresponding fluctuations.

4. INTENSITY: THE VARIOUS ASPECTS

Instantaneous intensity is defined by Eq. (1). From Eq. 10, it is immediate that its ensemble average is the integral of the spectrum. What the detector senses is the time integrated intensity I defined by Eq. (2). Because of the normalization condition, Eq. (3), it has the same ensemble average. But variance[9] and autocovariance significantly differ between the two kinds of intensities.

4.1. Variance and covariance of instantaneous intensity

We first consider instantaneous intensity. Its covariance is easily calculated only under the circular gaussian statistics assumption[10]. The gaussian character follows from the central limit theorem applied to the many uncorrelated atomic scale species that emit light in an mutually independent way in spontaneous emission processes. Circularity follows from the absence of privileged instants in time. The gaussian moment theorem then yields:

$$E\left[i(t_1)i(t_2)\right]=\left|E\left[U^*(t_1)U(t_2)\right]\right|^2+E\left[\left|U(t_1)\right|^2\right]E\left[\left|U(t_2)\right|^2\right] \tag{11}$$

whence the covariance of instantaneous intensity and, from Eq. (10), its power spectrum:

$$\mathrm{Cov}_{ii}(t_2-t_1)=\left|\mathrm{Cov}_{UU}(t_2-t_1)\right|^2 \tag{12}$$

and $S_i(\nu)=S_U\otimes S_U(\nu)$ (13)

From Eq. (12), it follows that the typical time variation scale of the instantaneous intensity is the coherence time, and that the instantaneous intensity power spectrum is directly related to the ordinary (complex amplitude) power spectrum : if the latter spans interval $[\nu_1;\nu_2]$, then the former spans $[\nu_1-\nu_2;\nu_2-\nu_1]$ with, as any autocorrelation function, a maximum at frequency zero.

The variance and signal to noise ratio of instantaneous intensity in turn follow:

$$\sigma_i^2=\mathrm{Cov}_{ii}(0)=\int_{\mathbb{R}}S_i(\nu)d\nu=\left(\int_{\mathbb{R}}S_U(\nu)d\nu\right)^2 \tag{14}$$

and $s/n_i=\dfrac{E(i)}{\sigma_i}=1$. (15)

4.2. Variance and covariance of detected intensity

From definition Eq. (3) and the stochastic process filtering theorem Eq. (A11), the power spectrum of the intensity as "seen" by the detector is

$$S_I=S_i\left|\tilde{R}\right|^2=(S_U\otimes S_U)\left|\tilde{R}\right|^2, \tag{16}$$

and the simplest form of the intensity correlation is the inverse Fourier Transform of Eq. (16). Frequencies present in the fluctuations of intensity I are restricted to those passing through the low pass filter of the detector response. Their order of magnitude is the response time T.

The signal to noise ratio follows as

$$s/n_I=\frac{\mathrm{E}(I)}{\left(\int_{\mathbb{R}}S_I(\nu)d\nu\right)^{1/2}}=\frac{\int_{\mathbb{R}}S_U(\nu)d\nu}{\left(\int_{\mathbb{R}}(S_U\otimes S_U)(\nu)\left|\tilde{R}(\nu)\right|^2d\nu\right)^{1/2}}. \tag{17}$$

As was stressed in section 2.3, response time is quite often very slow compared to the source coherence time, i.e. to the inverse spectral width of the source. In this limiting case, Eq. (17) can be simplified by introducing the following definitions for the average spectrum value S_a, the spectral width $\Delta\nu$, and the time response T:

$$\int_{\mathbb{R}} S_U(\nu)\,d\nu = S_a\,\Delta\nu \tag{18}$$

$$\int_{\mathbb{R}} S_U^2(\nu)\,d\nu = S_a^2\,\Delta\nu \tag{19}$$

$$\int_{\mathbb{R}} R^2(t)\,dt = \frac{1}{T} \tag{20}$$

Note that these definitions are fairly arbitrary and do not necessarily coincide with the definition of T given elsewhere in this chapter, but give reasonable orders of magnitudes.

With these in mind, and simplifying the denominator of Eq. (17) as follows based on the assumption that function S_U is much broader than function $\tilde{R}$ so that $\tilde{R}(\nu)$ vanishes before $S_U(\mu+\nu)$ significantly differs from $S_U(\mu)$:

$$\iint_{\mathbb{R}^2} S_U(\mu)S_U(\mu+\nu)\left|\tilde{R}(\nu)\right|^2 d\mu d\nu \simeq \int_{\mathbb{R}} S_U^2(\mu)\,d\mu \int_{\mathbb{R}} \left|\tilde{R}(\nu)\right|^2 d\nu \tag{21}$$

Eq. (17) reduces to

$$s/n_I = \frac{S_a\,\Delta\nu}{\sqrt{S_a^2\,\frac{\Delta\nu}{T}}} = \sqrt{T\,\Delta\nu} \tag{22}$$

or, in words: the classical detection noise signal to noise ratio is the square root of the number of correlation time intervals in the detector response time, which is also the square root of the number of independent samples of instantaneous intensity in one intensity measurement by the detector. Evidence of such behavior is provided by the numerical simulations shown in Figs. 1 and 2.

4.3 Numerical example:

Eq. (17) can also be carried further analytically in the case of the decreasing exponential detector response model of Eq. (4) and a flat power spectrum

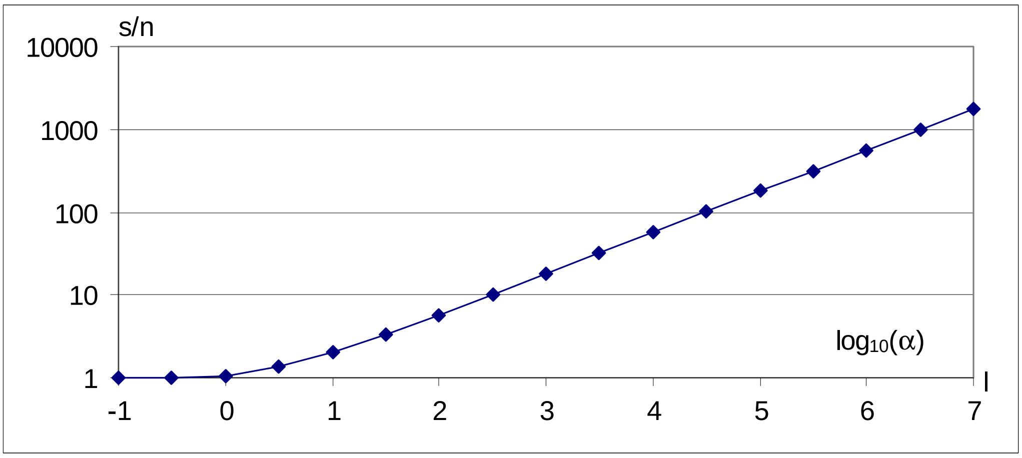


Figure 6. Measured intensity signal to noise ratio versus decimal logarithm of parameter α, which is essentially the number of coherence time intervals during the detector integration time.

$$S_U(\nu) = S_a \operatorname{rect}\frac{\nu - \nu_o}{\Delta\nu} \tag{23}$$

where rect is the unit height, unit width, even rectangle function. Straightforward calculations lead to

$$s/n_I = \left(\frac{2\arctan\alpha}{\alpha} - \frac{\ln\left(1+\alpha^2\right)}{\alpha^2}\right)^{-1/2} \quad \text{with } \alpha = 2\pi T\ \Delta\nu \tag{24}$$

Fig. (6) illustrates the result. With present technology, all orders of magnitude are possible: assuming flat spectra in the sense of Eq. (23), a nearly unit signal to noise ratio is reached by an ultrafast (T=10ps) detector exposed to a 0.1 nm wide spectral line, while the eye (T=0,1s) looking at a sunlight illuminated scene ($\Delta\lambda$=300 nm) senses strictly no noise with a signal to noise ratio as high as 250000. Nevertheless, by far the most common situation is a fairly high signal to noise ratio, which explains that classical detection noise is a little known phenomenon and far from the student's intuition.

5. THE LAW OF INTERFERENCE AND THE CORRESPONDING FLUCTUATIONS

After this review of classical detection noise, we can now come back to our initial problem of giving a statistical account of the reason why "two different frequencies do not interfere" in thermal light. Because we are interested in temporal coherence, we consider the two beam interferometer already mentioned. In addition we now assume that the source is small enough to effectively act as one source point. In that case, the complex amplitude U_μ that illuminates the interferometer is split into two parts that travel through the two arms and reach the detector with a travel time difference θ. Because of the assumption that the source is a point and assuming moreover that no dispersion effects occur, this time delay is the only difference between the two complex amplitudes that emerge from the source. Assuming in addition, for simplicity, that the two arms carry identical amounts of power and ignoring unessential constants, the detector receives

$$U_{1\mu}(t) = U_\mu(t) + U_\mu(t+\theta) \tag{25}$$

In the following, subscript 1 refers to U_1 and its statistical properties, i.e. to the output of the interferometer. Eq. (25) straightforwardly implies

$$\operatorname{Cov}_{U_1U_1}(\tau) = 2\operatorname{Cov}_{UU}(\tau) + \operatorname{Cov}_{UU}(\tau+\theta) + \operatorname{Cov}_{UU}(\tau-\theta) \tag{26}$$

and $S_{U1}(\nu) = S_U(\nu)\left(1+\cos 2\pi\nu\theta\right)$ (27)

with, as expected, as the chief consequence, an average intensity at the output of the interferometer

$$E(i_1) = E(I_1) = \operatorname{Cov}_{U_1U_1}(0) = \int_{\mathbb{R}} S_U(\nu)\left(1+\cos 2\pi\nu\theta\right)d\nu \tag{28}$$

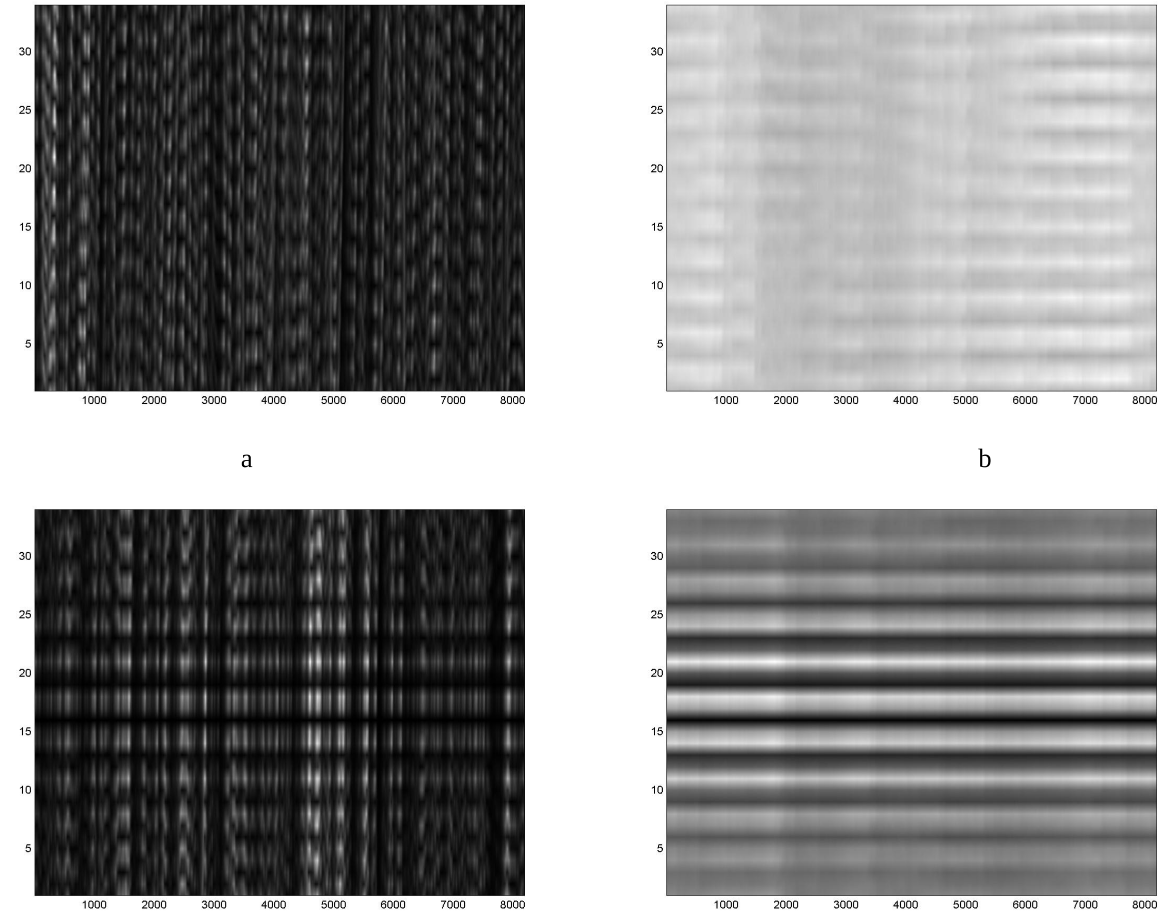
which is the sum of the monochromatic interference terms of all components frequencies, weighted by their spectral density.

The whole development of section 4 applies directly to random process U_1. In particular, in usual interferometers, the time delay between the two arms is small compared to the detector response time T (some modern very large interferometers are exceptions). Therefore, just without the interferometer, the measured intensity suffers small fluctuations whose characteristic autocorrelation time is given by T, the detector response time and not by the source or by the interferometer. The signal to noise ratio is given by Eq. (17) applied to spectral density S_{U1}, i.e.

$$s/n_{I1} = \frac{\mathrm{E}(I_1)}{\left(\int_{\mathbb{R}} S_{I1}(\nu)\,d\nu\right)^{1/2}} = \frac{\int_{\mathbb{R}} S_{U1}(\nu)\,d\nu}{\left(\int_{\mathbb{R}} \left(S_{U1}\otimes S_{U1}\right)(\nu)\left|\tilde{R}(\nu)\right|^2 d\nu\right)^{1/2}} \tag{29}$$

While the exact signal to noise ratio value is affected by the interferences through their effect on the spectral density, the order of magnitude is unchanged. This is how we reach our main conclusion: according to Eq. (28), on average two different frequencies never interfere, they merely add up their contributions. According to Eq. (29) and with the exception of very fast detectors, fluctuations exist, but are usually too small to be even mentioned. Indeed, accurate measurements with fast detectors do reach the classical detection noise and can account for the influence of the interferometer on measured intensity fluctuations[11,12].

Figure 7 summarizes the cases illustrated by the previous simulations. This time, intensity as detected by the detector is plotted against time (horizontal axis) and against path difference (vertical axis), thus simulating a screen visualizing interferences. 7a and 7b show the case of two difference sources emitting the same spectrum, with 7b corresponding to a much longer response time and clearly indicating blurring of the fringes. 7c and 7d correspond to the spatially coherent case of one single source point; fringes do not move, time averaging does not blur them out.



a b c d

Figure 7. Horizontal: time. Vertical: path difference on interferogram. a and c, fast detector (T=3 pseudo-periods); b and d, slower detector (T=300 pseudo-periods); a and b: two different source points; c and d: single source point.

Before concluding, it is appropriate that we stress once more that we have covered thermal light only. Laser light differs from thermal light in many respect, and in particular the mutual phase of laser modes that can operate simultaneously are correlated. An extreme case of such a correlation is mode locking, that allows to reach extremely short pulse duration by a perfect phase synchronization of many modes. In such a case, in a way, different frequencies do interfere. However, this is a different case of interferences, which directly results in observable beat phenomena in the time domain while the more common case discussed above operates on time integrated signals and often results in fringes in the space domain; whether one still wants to use the word "interferences" to describe the latter situation will be left open.

6. CONCLUSION

Stationary covariance stochastic processes are the appropriate tool for a clear description of interference phenomena with thermal light sources. Their introduction as a prerequisite to advanced courses on coherence that will finalize the student learning process on classical fields interferences is an obvious need. The close relation that exists between the classical detection noise and the usual law that "two different frequencies do not interfere" would seem to deserve more attention in the teaching of these effects.

References

[1] J.W. Goodman, *Statistical Optics*, John Wiley & Sons, New York (1985).
[2] M. Françon and S. Slansky, *Cohérence en Optique*, Editions du Centre National de la Recherche Scientifique, Paris, 1965, chapter 1.
[3] A.W. Lohmann's lecture notes on *Optical Information Processing*, Erlangen edition (1978), part 2, § 3.
[4] F.L. Pedrotti and L.S. Pedrotti, *Introduction to Optics*, Prentice Hall, Upper Saddle River, New Jersey, 1993, section 12-4.
[5] See e.g. section 4.2.1 of Ref. 1.
[6] the vocabulary seems to imply that it is independent of the detector, which indeed is not quite true as will be seen in the present work.
[7] the product of two singular distributions, such as the square of the Dirac delta, is in general not defined.
[8] P. Chavel and J.M. Jonathan, "A statistical introduction to temporal coherence effects in interferometers", to appear in SPIE Press Volume "Tribute to Adolf Lohmann", H.J. Caulfield, editor, 2001.
[9] For a thorough coverage, see section 6 of Ref. 1.
[10] See sections 3.9 and 4.2 of Ref. 1.
[11] Y. Weissman, "Optical noise coherence effects in a Mach-Zehnder system", *J. Lightwave Technol.* **14** (5) 888-893 (1996).
[12] K.B. Hill et al., "Noise and information in interferometric cross correlators", *Appl. Opt.* **36**(17) 3948-3958 (1997).