An analytical Prj2CR covariance estimation method for iterative CT reconstruction algorithms

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ABSTRACT

Image mean and covariance required for a model observer are usually calculated by the statistical method using image samples, which is hard to acquire in reality. Although some analytical methods are proposed to estimate image covariance from a single projection, these methods are of high computational cost for large-dimensional images (e.g., 512×512), and images of large dimension are commonly required. Considering the covariance used for a model observer is the covariance of the channel response, whose dimension is much smaller than the image covariance, we aim to obtain the covariance of small-dimensional channel response directly from its projection. Channel filters are applied to the analytical projection to image (Prj2Img) covariance estimation method to derive the analytical projection to channel response (Prj2CR) covariance estimation method, which successfully reduces the computational cost and connects the covariance of projection and channel response. In addition, a transition matrix is introduced in Prj2CR method to stabilize the connections. The transition matrix mainly depends on channel filters, not the system, phantom, and reconstruction algorithm, which means it can be calibrated by small-dimensional reconstructions and then applied to any situation with a same channel filter. We validate the feasibility and utility of the proposed Prj2CR method by simulations. 128×128 reconstructions from qGGMRF-WLS are adopted for calibration, while 512×512 reconstructions are used for validation. SNR of CHO is chosen as the figure of merit for performance evaluation, and the covariance estimated by 290 image samples are used as the reference. Results show that the SNR by the Prj2CR method is within 95% confidence interval of the SNR* by 290 image samples, indicating that the proposed method accords with statistical method. The Prj2CR method may be beneficial for subjective image quality assessment since it only needs a single sample of projection and has low computational cost.

Keywords: Model observer, covariance estimation, iterative CT reconstruction

1. INTRODUCTION

Model observers are used to mimic human observers. However, they require the knowledge of image mean and covariance, which is difficult to achieve in reality. Analytical projection to image (Prj2Img) covariance estimation methods are proposed to estimate image covariance from a single projection for commonly used iterative CT reconstruction algorithms. Iteration-based and fixed-point methods are two ways to analytically estimate covariance from projection to image. For iteration-based methods, the covariance estimation is updated with iteration formula[1]. For fixed-point methods, the covariance estimation is derived from the converged point of an objective function[2]. Li[3] et.al. study the difference and consistence of these two methods. Meanwhile, analytical Prj2Img methods for iterative reconstruction with a quadratic regularization is studied by Schmitt[4] et.al., while that with a non-quadratic regularization is studied by Sánchez[5]. Although analytical Prj2Img methods can yield reasonable covariance estimations, they are computationally expensive for large-dimensional images (pixels > 128×128). Usually, images with larger size, e.g., 512×512, are required for practical use. Fessler[6] et.al. give a fast variance estimation method for the quadratic penalized weighted least square (WLS) algorithm. Fast covariance estimation methods have not been studied yet.

In fact, the covariance used for model observers is the covariance of the channel response[7] that has a much smaller dimension than its image covariance matrix. Therefore, we target on estimating the covariance of low-dimensional channel response from high-dimensional projection. The proposed projection to channel response (Prj2CR) covariance estimation method is derived from the Prj2Img method. We apply the proposed method to a widely used channelized Hotelling observer (CHO) with large-dimensional image inputs for validation.

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7th International Conference on Image Formation in X-Ray Computed Tomography,
edited by Joseph Webster Stayman, Proc. of SPIE Vol. 12304, 123041J

2. METHODS

In this section, we briefly introduce the analytical Prj2Img covariance estimation method for non-quadratic penalized WLS studied in our previous work[8], then describe the CHO, and finally deduce the proposed analytical Prj2CR covariance estimation method in detail.

2.1 Analytical Prj2Img covariance estimation method

The cost function $\Phi(\cdot)$ of a penalized WLS reconstruction can be expressed as:

$$\hat{\boldsymbol{\mu}} = \arg\min_{\boldsymbol{\mu}} \Phi(\boldsymbol{\mu}) = \arg\min_{\boldsymbol{\mu}} \frac{1}{2}\left\|\mathbf{H}\boldsymbol{\mu} - \mathbf{p}\right\|_{\mathbf{W}}^2 + \alpha R(\boldsymbol{\mu}) \tag{1}$$

where $\hat{\boldsymbol{\mu}} \in \mathbb{R}^{N\times 1}$ is the linear attenuation image reconstructed from its projection $\mathbf{p} \in \mathbb{R}^{M\times 1}$. $\mathbf{H} \in \mathbb{R}^{M\times N}$ denotes the system matrix and $\mathbf{W} \in \mathbb{R}^{M\times M}$ the noise model with $\mathbf{W}_{mm} = \exp(-p_m)$. $R(\cdot)$ is the penalty function and α the penalty parameter. The fixed-point method makes use of convergence condition:

$$\left.\frac{\partial \Phi(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}\right|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}} = \mathbf{0} \tag{2}$$

Plugging Eq. (1) into Eq. (2) and finding its covariance, we have:

$$(\mathbf{H}^T\mathbf{W}\mathbf{H} + \alpha\mathbf{A})\mathrm{Cov}(\hat{\boldsymbol{\mu}})(\mathbf{H}^T\mathbf{W}\mathbf{H} + \alpha\mathbf{A}^T) \simeq \mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H} \tag{3}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is a coefficient matrix approximates $\nabla R(\hat{\boldsymbol{\mu}})$:

$$\nabla R(\hat{\boldsymbol{\mu}}) \simeq \mathbf{A}\hat{\boldsymbol{\mu}} + \mathbf{c} \tag{4}$$

Our previous work proposed the linear based method (LAM) with $\nabla R(\hat{\boldsymbol{\mu}}) = \mathbf{L}(\hat{\boldsymbol{\mu}})\hat{\boldsymbol{\mu}} \simeq \mathbf{L}(\bar{\hat{\boldsymbol{\mu}}})\hat{\boldsymbol{\mu}}$, thereby estimates the covariance as:

$$\left(\mathbf{H}^T\mathbf{W}\mathbf{H} + \alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}})\right)\mathrm{Cov}(\hat{\boldsymbol{\mu}})\left(\mathbf{H}^T\mathbf{W}\mathbf{H} + \alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}})\right) \simeq \mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H} \tag{5}$$

here, $\bar{\hat{\boldsymbol{\mu}}}$ is the expectation of $\hat{\boldsymbol{\mu}}$. We adopt the WLS penalized with the total variance (TV-WLS) as well as qth generalized Gaussian Markov random field (qGGMRF-WLS) as two representatives in this work.

2.2 Channelized Hotelling observer (CHO)

A CHO is calculated as:

$$\lambda = [\mathbf{S}_c^{-1}(\bar{\hat{\boldsymbol{\mu}}}_{sc} - \bar{\hat{\boldsymbol{\mu}}}_{bc})]^T \hat{\boldsymbol{\mu}}_c \equiv \boldsymbol{\omega}^T \hat{\boldsymbol{\mu}}_c \tag{6}$$

where λ denotes the decision variable and $\boldsymbol{\omega}$ the template of CHO. The image of channel response is denoted by a subscript c:

$$\hat{\boldsymbol{\mu}}_c = \mathbf{V}\hat{\boldsymbol{\mu}} \tag{7}$$

with $\mathbf{V} \in \mathbb{R}^{C\times N}$ being the channel matrix consists of C channel profiles and $C \ll N$. Meanwhile, the $\bar{\hat{\boldsymbol{\mu}}}_{sc}$ and $\bar{\hat{\boldsymbol{\mu}}}_{bc}$ in template $\boldsymbol{\omega}$ represent the mean of signal present and signal absent images respectively. The intra-class channel scatter matrix $\mathbf{S}_c = (\mathbf{K}_{sc} + \mathbf{K}_{bc})/2$, where the channel covariance $\mathbf{K}_{sc} = \mathbf{V}\mathbf{K}_s\mathbf{V}^T$ and $\mathbf{K}_{bc} = \mathbf{V}\mathbf{K}_b\mathbf{V}^T$ with $\mathbf{K} = \mathrm{cov}(\hat{\boldsymbol{\mu}})$ being the image covariance. For a given threshold t, if the decision variable satisfies $\lambda > t$, we consider the target image to contain the signal; otherwise, we consider it not. Meanwhile, signal-to-noise ratio[9] (SNR) is adopted to measure the performance of a CHO:

$$\mathrm{SNR} = \sqrt{(\bar{\boldsymbol{\mu}}_{sc} - \bar{\boldsymbol{\mu}}_{bc})^T \mathbf{S}_c^{-1} (\bar{\boldsymbol{\mu}}_{sc} - \bar{\boldsymbol{\mu}}_{bc})} \tag{8}$$

2.3 Analytical Prj2CR covariance estimation method

Note that the analytical Prj2Img method in Eq. (5) involves a matrix inverse operation, which leads to high computational cost when the dimension of the reconstructed image is high. As described in section 2.2, the CHO only requires a small-dimensional covariance of channel response, where $C \ll N$. Therefore, we construct a relationship between the covariance of channel response $\mathbf{K}_c = \mathbf{V}\mathbf{K}\mathbf{V}^T$ and the covariance of corresponding projection $\mathrm{Cov}(\mathbf{p})$:

$$\begin{aligned} &\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathrm{Cov}(\hat{\boldsymbol{\mu}})(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}})^T)\mathbf{V}^T \\ &= \mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathbf{V}^T\mathbf{X}^*[\mathbf{V}\mathbf{K}\mathbf{V}^T]\mathbf{X}^{*T}\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}})^T)\mathbf{V}^T \\ &\simeq \mathbf{V}\mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H}\mathbf{V}^T \end{aligned} \tag{9}$$

where the transition matrix $\mathbf{X}^* = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T$. Note that matrix $\mathbf{V}^T\mathbf{V}$ is not full rank, and hence it is irreversible. Hypothesis there exists an invertible transition matrix $\mathbf{X} \in \mathbb{R}^{C\times C}$ that satisfies Eq.(9):

$$\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathbf{V}^T\mathbf{X}\mathbf{K}_c\mathbf{X}^T\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}})^T)\mathbf{V}^T \simeq \mathbf{V}\mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H}\mathbf{V}^T \tag{10}$$

Both covariance of channel response $\mathbf{K}_c$ and transition matrix $\mathbf{X}$ are unknown. In theory, $\mathbf{X}$ only depends on channels, not systems, phantoms, and reconstruction algorithms. Thus, we can calibrate $\mathbf{X}$ using known $\mathbf{K}_c$ of low dimension. $\mathbf{X}$ calibration is expressed as an optimization problem:

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \frac{1}{2}\left\|\mathbf{Q}\mathbf{K}_c\mathbf{Q}^T - \mathbf{V}\mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H}\mathbf{V}^T\right\|_2^2 \tag{11}$$

where $\mathbf{Q} = [\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathbf{V}^T]\mathbf{X}$. We split Eq. (11) into two sub-problems to make it easier to get a reasonable solution:

$$\hat{\mathbf{Q}} = \arg\min_{\mathbf{Q}} \frac{1}{2}\left\|\mathbf{Q}\mathbf{K}_c\mathbf{Q}^T - \mathbf{V}\mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H}\mathbf{V}^T\right\|_2^2 \tag{12.1}$$

$$\hat{\mathbf{X}} = [\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathbf{V}^T]^{-1}\hat{\mathbf{Q}} \tag{12.2}$$

Since $\mathbf{X}$ is an underdetermined matrix, the optimization problem in Eq. (12) is locally convergent. Thus, we choose starting points as:

$$\mathbf{Q}^{(0)} = [\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{A})\mathbf{V}^T][(\mathbf{V}^T)^?\mathbf{V}^\dagger], \quad \mathbf{X}^{(0)} = \mathbf{I}$$

with $(\cdot)^\dagger$ being the Moore-Penrose generalized inverse operation. Plugging the calibrated $\hat{\mathbf{X}}$ into Eq. (10), we can finally obtain the covariance estimation $\hat{\mathbf{K}}_c$ under arbitrary conditions:

$$\hat{\mathbf{K}}_c \simeq (\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathbf{V}^T\hat{\mathbf{X}})^{-1}\mathbf{V}\mathbf{H}^T\mathbf{W}\mathrm{Cov}(\mathbf{p})\mathbf{W}\mathbf{H}\mathbf{V}^T[(\mathbf{V}(\mathbf{H}^T\mathbf{W}\mathbf{H}+\alpha\mathbf{L}(\bar{\hat{\boldsymbol{\mu}}}))\mathbf{V}^T\hat{\mathbf{X}})^T]^{-1} \tag{13}$$

3. EXPERIMENTS

Gabor function is used for CHO in this work:

$$v(x,y) = \exp[-4\ln 2((x-x_0)^2+(y-y_0)^2)/\omega_s^2]\cdot\cos[2\pi f_c((x-x_0)\cos\theta+(y-y_0)\sin\theta)+\beta] \tag{14}$$

here the parameter configuration of Gabor channels is similar to that used in Leng's work[10], where the channel width $\omega_s = 56.48, 28.24, 14.12, 7.06$, the channel frequency $f_c = \frac{3}{128}, \frac{3}{64}, \frac{3}{32}, \frac{3}{16}$, the orientation $\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$, and the phase $\beta = 0, \pi/2$.

Table 1. Reconstruction parameters for calibration and validation of Prj2CR covariance estimation method.

Reconstruction Parameters	Calibration	Validation
Source to origin distance (mm)	200	595
Origin to detector distance (mm)	200	490.6
Size of detector bins (mm)	0.5	1.2858
Projection dimension	240×360	736×360
Size of image dimension	0.3	0.7422
Image dimension	128×128	512×512
Incident photons	$10^7/10^5/5\times103$	$10^6/5\times10^5/3\times10^5$
Reconstruction algorithms	qGGMRF-WLS	TV-WLS/qGGMRF-WLS
Image covariance estimation method	Prj2Img	/
Covariance of channel response estimation method	/	Prj2CR

Phantoms for calibration and validation are generated from the Grand challenge dataset of Mayo clinic respectively. For calibration, images of size 128×128 are reconstructed by qGGMRF-WLS with system defined in Table 1. Since $\mathbf{X}$ is underdetermined, transition matrix $\mathbf{X}$ of various noise levels are averaged for stability. For validation, large-dimensional images are reconstructed by both TV-WLS and qGGMRF-WLS with three noise levels as is shown in Table 1. Covariance estimated by numerous image samples is used as reference:

$$\mathbf{K}_c^* = \mathbf{V}\mathrm{Cov}^*(\hat{\boldsymbol{\mu}})\mathbf{V}^T = \mathbf{V}[\frac{1}{K-1}\sum_{k=1}^{K}(\hat{\boldsymbol{\mu}}_k - \bar{\hat{\boldsymbol{\mu}}})^T(\hat{\boldsymbol{\mu}}_k - \bar{\hat{\boldsymbol{\mu}}})]\mathbf{V}^T \tag{15}$$

with K being number of image samples. To better evaluate the performance of proposed Prj2CR method, we choose SNR of CHO as the figure of merit according to Eq. (8):

$$\mathrm{SNR} = \sqrt{(\bar{\boldsymbol{\mu}}_{sc} - \bar{\boldsymbol{\mu}}_{bc})^T \hat{\mathbf{K}}_c^{-1}(\bar{\boldsymbol{\mu}}_{sc} - \bar{\boldsymbol{\mu}}_{bc})}$$
$$\mathrm{SNR}^* = \sqrt{(\bar{\boldsymbol{\mu}}_{sc} - \bar{\boldsymbol{\mu}}_{bc})^T (\mathbf{K}_c^*)^{-1}(\bar{\boldsymbol{\mu}}_{sc} - \bar{\boldsymbol{\mu}}_{bc})}$$

where SNR is calculated from the covariance estimated by Prj2CR method, while SNR^* is calculated from the covariance estimated by 290 image samples. Besides, we take the ground-truth as the image mean to minimize other influence factors, i.e., $\bar{\boldsymbol{\mu}}_{sc} = \bar{\boldsymbol{\mu}}_{sc}^{\mathrm{groundtruth}}$ and $\bar{\boldsymbol{\mu}}_{bc} = \bar{\boldsymbol{\mu}}_{bc}^{\mathrm{groundtruth}}$.

4. RESULTS

SNR performances are displayed in Figure 1. The SNR calculated by the proposed Prj2CR method is basically within the 95% confidence interval of the SNR^* calculated by image samples. For qGGMRF-WLS reconstructions in case of 3×10^5 incident photons, the SNR by the Prj2CR method underestimates SNR^* by image samples.

5. DISCUSSION AND CONCLUSION

We proposed an analytical Prj2CR covariance method in this work, which can estimate the covariance of channel response directly from a single projection. The proposed method solves the problem of high-computational cost of Prj2Img method, and enables the covariance estimation of high-dimensional reconstructions. In this work, we introduce an invertible transition matrix to connect covariance of low-dimensional channel response and high-dimensional projection. Meanwhile, we calibrate the transition matrix to make the proposed method works for different systems, phantoms and reconstruction algorithms. The covariance of channel response estimated by Prj2CR method is comparable to that by 290 image samples.

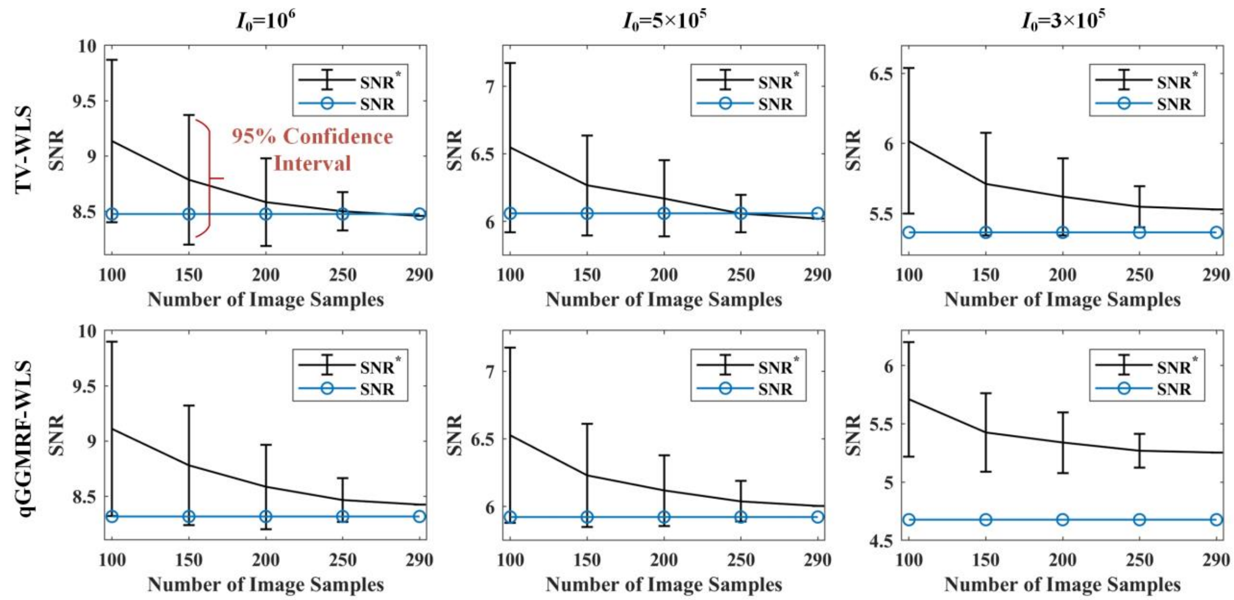

Figure 1. CHO performance of Prj2CR method and image samples.

6. ACKNOWLEGMENTS

This work is supported by National Natural Science Foundation of China (Grant No. 62031020 and 61771279).

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