X-ray CT Data Completeness Condition for Sets of Arbitrary Projections

Gabriel Herl[a], Andreas Maier[b], and Simon Zabler[a]

[a]Deggendorf Institute of Technology, Dieter-Görlitz-Platz 1, 94469 Deggendorf, Germany
[b]Department of Pattern Recognition, Friedrich-Alexander University Erlangen-Nürnberg, Martensstraße 3, 91058 Erlangen, Germany

ABSTRACT

X-ray tomography reconstruction requires a set of projections that provides sufficient information for the examined region. Commonly, to ensure mathematically complete reconstruction, first, a continuous curve (trajectory) that fulfils the Tuy conditions is chosen. Second, this curve is sampled based on the Nyquist-Shannon sampling theorem. This two-step approach is efficient for most standard X-ray tomography scanning scenarios. For agile X-ray tomography systems, e.g. robot-supported computed tomography systems, choosing a set of projections based on a continuous curve is often not useful. Instead, sets of projections from arbitrary views might be necessary.

This work combines the Tuy-Smith condition with conclusions from the Nyquist-Shannon sampling theorem. In particular, the maximal pixel size and a requirement for the arrangement of projections are formulated depending on the smallest relevant object feature and integrated into the Tuy-Smith condition. We derive a comprehensive condition for data completeness that can assess the completeness of any set of arbitrary projections, e.g. for complex scanning scenarios with robot-supported X-ray tomography systems.

Keywords: Data completeness condition, arbitrary scan geometry, robot-supported CT, Tuy conditions, Nyquist-Shannon sampling theorem

1. INTRODUCTION

X-ray computed tomography (CT) allows digitisation of three-dimensional inner and outer structures for many applications in medicine[1] and in industry.[2] In a standard industrial CT, the examined object is placed on a turntable between an X-ray source and a detector. By rotating the object, projection views from a circular trajectory around the object are generated.

In addition to these classical systems, twin robotic CT systems have been developed where the source and the detector are mounted on individual robots. This setup allows agile movement of source and detector in order to create projections from arbitrary views around the object. This agility of robot-supported CT systems enables new, more complex trajectories which can be utilised in numerous ways, e.g. to scan large-scale objects,[3] to reduce metal artefacts[4,5] and to reduce the scan time.[6]

However, this agility of robot-supported CT systems complicates the CT scan process. In order to enable a true, mathematically complete reconstruction, it has to be ensured that the available projections generate sufficient information. In the standard CT process, first, the CT user chooses a continuous curve (commonly a circle or a helix) around a region of interest. Second, by choosing the number of projections, this continuous curve is sampled into individual, equidistantly placed views for the generation of individual projections.

A two-step approach is performed to ensure that the resulting set of projections provides sufficient information for mathematically complete reconstruction. First, the Tuy-Smith condition[7] assesses whether the continuous trajectory can provide enough information. Second, the sampling is validated based on the Nyquist-Shannon sampling theorem. This two-step process is efficient and sufficient for most scenarios for standard CT systems. However, especially when using agile CT systems, this process has weaknesses. The two-step approach requires

Send correspondence to Gabriel Herl, e-mail: gabriel.herl@th-deg.de

7th International Conference on Image Formation in X-Ray Computed Tomography,
edited by Joseph Webster Stayman, Proc. of SPIE Vol. 12304, 123040C

a continuous curve as a base trajectory. However, in many scenarios, the requirement of a continuous curve is impractical. Fig. 1 shows an example from[4] of two sets of views for a region of interest scan of a defect in a motorcycle. A continuous curve would not be a useful basis for both sets of views.

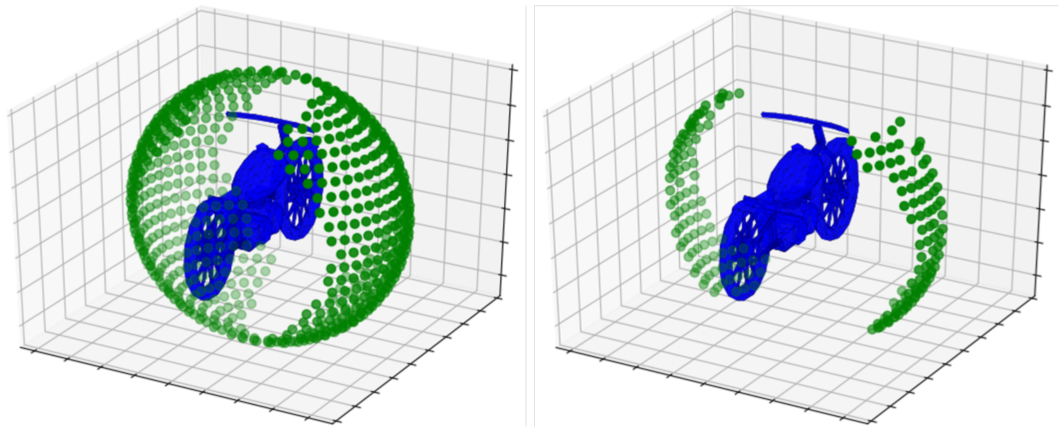

Figure 1. Visualisation of two sets of views on a motorcycle. The green dots represent individual source positions.

This work extends the Tuy-Smith condition with the Nyquist-Shannon sampling theorem to create a single condition that indicates data completeness for a mathematically complete reconstruction. For the first time, this condition allows the assessment of data completeness for arbitrary sets of views.

2. STATE OF THE ART OF CT DATA COMPLETENESS

The X-ray attenuation process can be depicted as the Radon transform which maps every hyperplane of the spatial domain to a point in the Radon space. Let $\mathbf{f}$ be the density function of the examined object, $\mathbf{u}$ be the normal vector of a hyperplane and s be the distance of this plane to the origin. The Radon transform can be written as

$$\mathbf{Rf}(\mathbf{u},\mathbf{s}) = \int_{\mathbf{x}\cdot\mathbf{u}=s} \mathbf{f}(\mathbf{x})dx. \tag{1}$$

In 1917,[8] Radon proved that if $\mathbf{f}(\mathbf{u},\mathbf{s})$ is continuous, there exists a unique inverse. Therefore if the Radon space is continuous and known, complete CT reconstruction can be ensured. However, it is not trivial whether a specific trajectory or a specific set of arbitrary X-ray projections can be applied to measure the Radon space sufficiently.

2.1 Trajectory Requirements

In 1983,[9] Tuy published data completeness conditions for continuous curves that ensure mathematically correct reconstruction. Following Tuy, a curve is a continuous function $\Phi : \Lambda \to \mathbb{R}^3$ where Λ is an interval in $\mathbb{R}$. Let $\mathbb{S} \subset \mathbb{R}^3$ be the unit sphere and $\Omega \subset \mathbb{R}^3$ be a compact region that contains the complete object with density function $\mathbf{f}$.

Tuy implicitly used several assumptions about the measuring and reconstruction processes to ensure mathematically correct reconstruction. This includes the following two (unrealistic) assumptions:

1. **Detector resolution assumption:** Tuy assumes that the exact position of impact of each measured photon is known. This corresponds to a detector with infinitely small pixels.
2. **Source assumption:** Tuy assumes that X-rays are emitted at every position of the trajectory of the source. This assumption corresponds to an infinite number of projections on the trajectory.

Tuy conditions
Using assumptions 1 and 2, an object in region Ω can be reconstructed from projection data generated on a curve Φ if the following three conditions are valid:

1. *The curve Φ is outside of the region Ω.*
2. *The curve Φ is bounded, continuous and almost everywhere differentiable.*
3. *For all $(\mathbf{x}, \mathbf{u})$ in $\Omega \times \mathbb{S}$, there exists $\lambda \in \Lambda$, such that $\mathbf{x}^\top \mathbf{u} = \Phi(\lambda)^\top \mathbf{u}$ and $\Phi'(\lambda)^\top \mathbf{u} \neq 0$.*

In 1985,[7] Smith proved that Tuy's third condition is sufficient:

Tuy-Smith condition
Using assumptions 1 and 2, an object in region Ω can be reconstructed from projection data generated on a curve Φ if, for all $(\mathbf{x}, \mathbf{u})$ in $\Omega \times \mathbb{S}$, there exists $\lambda \in \Lambda$, such that $\mathbf{x}^\top \mathbf{u} = \Phi(\lambda)^\top \mathbf{u}$.

Descriptively, the Tuy-Smith condition states that every plane through the region of interest must intersect the source trajectory. As every point in the Radon space corresponds to one plane in the spatial domain, the Tuy-Smith condition ensures a full sampling of the Radon space.

2.2 Sampling Requirements

As both assumptions, the detector resolution assumption and the source assumption of Section 2.1, are impossible in practice, the projections as well as the continuous trajectory need to be sampled. To nevertheless ensure sufficient information, the Nyquist-Shannon sampling theorem can be applied.

A detailed derivation of the maximal pixel size and minimal number of projections is presented by Buzug.[10] Let k be the minimal magnification factor and $f_{\min}$ be the smallest relevant feature. Based on the Nyquist-Shannon sampling theorem, the maximal pixel size $\triangle\xi$ is given by

$$\triangle\xi < \frac{k}{2} f_{\min} \tag{2}$$

Let r be the radius of the measuring field. Following Buzug,[10] the maximal angular gap $\triangle\gamma$ between projections is given by

$$\triangle\gamma < \frac{f_{\min}}{2r}. \tag{3}$$

As an example for parallel-beam geometry, for an equiangular sampling of a semi-circular trajectory, the minimum number of projections follows by

$$n = \frac{\pi}{\triangle\gamma}. \tag{4}$$

3. NEW DATA COMPLETENESS CONDITION FOR SETS OF ARBITRARY VIEWS

To directly assess the data completeness of a set of arbitrary projections, we combine the Tuy-Smith condition with the presented conclusions of the Nyquist-Shannon sampling theorem. This means that, first, the pixel size and, second, the maximal angular gap of projections are integrated in the Tuy-Smith condition.

The maximal pixel size can directly be applied to create a realistic new assumption:

1b. **Adapted detector resolution assumption:** The detector has a maximal pixel size

$$\triangle\xi < \frac{k}{2} f_{\min}$$

based on the smallest magnification k of any of the used projections and the size $f_{\min}$ of the smallest feature that should be detectable.

To integrate the maximal angular gap into the Tuy-Smith condition, we extend the estimations of Maier *et al.*[11] We assume parallel-beam scanning geometry to allow more intuitive and straightforward phrasing. The conclusions remain for cone-beam CT. Let $\tilde{\mathbf{u}} \in \mathbb{S}$ be a normal vector that represents an arbitrary plane in Radon space and $\mathbf{D}$ the set of the normal vectors of all measured planes in Radon space. According to the Nyquist-Shannon sampling theorem, the angular distance between neighbouring planes in Radon space does not have to be zero, but only must be smaller than the specified maximum angular gap $\triangle\gamma$ of (3). This means, for any possible plane in Radon space, there has to be a measured plane that is tilted less than $\triangle\gamma$. The cosine angle between two vectors equals the inner product of the corresponding unit vectors. Thus, this condition can be written as

$$\forall \tilde{\mathbf{u}} \in \mathbb{S} \; \exists \mathbf{d} \in \mathbf{D} \; : \; |\cos^{-1}(\mathbf{d}^\top \tilde{\mathbf{u}})| \leq \triangle\gamma. \tag{5}$$

Let $\mathbf{u} \in \mathbb{S}$ be a vector perpendicular to vector $\tilde{\mathbf{u}}$. Vector $\tilde{\mathbf{u}}$ being tilted less than $\triangle\gamma$ according to vector $\mathbf{d}$ equals vector $\mathbf{u}$ being perpendicular to vector $\mathbf{d}$ apart from an angle $\triangle\gamma$. Two vectors are perpendicular apart from an angle $\triangle\gamma$ if $|\mathbf{d}^\top \mathbf{u}| \leq \sin(\triangle\gamma)$. Equation (5) thus is equivalent to

$$\forall \mathbf{u} \in \mathbb{S} \; \exists \mathbf{d} \in \mathbf{D} \; : \; |\mathbf{d}^\top \mathbf{u}| \leq \sin(\triangle\gamma). \tag{6}$$

The normal vector of a measured plane in Radon space corresponds to the directional vector of the projection that measured this plane in Radon space. Let $\mathbf{L}$ be the set of all source positions of a set of projections and $\mathbf{x} \in \Omega$ any point in the region of interest. Defining directional vectors of projections by $\mathbf{d}_{\mathrm{x,l}} := \frac{\mathbf{x}-\mathbf{l}}{||\mathbf{x}-\mathbf{l}||_2}$, Equation (6) can be written as

$$\forall (\mathbf{x}, \mathbf{u}) \in \boldsymbol{\Omega} \times \mathbb{S} \; \exists \mathbf{l} \in \mathbf{L} \; : \; |\mathbf{d}_{\mathrm{x,l}}^\top \mathbf{u}| \leq \sin(\triangle\gamma). \tag{7}$$

This equation ensures that sufficient projections for a complete reconstruction are available. In total, by integrating the conclusions of the Nyquist-Shannon sampling theorem (Assumption 1b and Equation (6)) into the Tuy-Smith condition, the following combined condition can be derived:

Data completeness condition for sets of arbitrary projections

An object in region $\boldsymbol{\Omega}$ can be reconstructed based on projections with corresponding source positions $\mathbf{L}$ if

1. *the maximal pixel size of the projections is $\triangle\xi < \frac{k}{2} f_{min}$ depending on the smallest magnification factor k and the smallest relevant feature f_{min},*
2. *for all $(\mathbf{x}, \mathbf{u}) \in \boldsymbol{\Omega} \times \mathbb{S}$, there exists a projection with corresponding source location $\mathbf{l} \in \mathbf{L}$ such that*

$$|\mathbf{d}_{x,l}^\top \mathbf{u}| \leq \sin(\triangle\gamma)$$

with $\triangle\gamma = \frac{f_{min}}{2r}$ as the maximal sufficient angular sampling rate depending on the radius r of the measuring field.

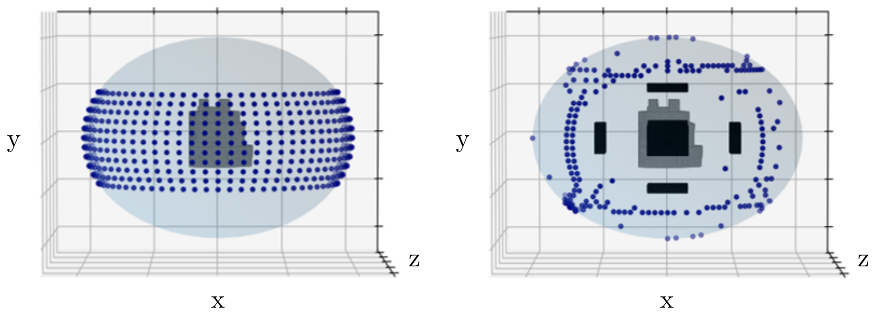

Figure 2. Two data complete sets of views: Visualisation of the source positions and a test specimen. The right scenario also contains highly attenuating metal blocks. The presented set of projections mostly avoids X-rays through these blocks, generating complete data for the reconstruction of the test specimen without metal disturbance.

4. EXAMPLES

As a short example, Fig. 2 shows two additional sets of views for a plastic test specimen that all fulfil the presented data completeness condition. The right image contains highly attenuating metal blocks. This example demonstrates that our condition can be applied to assess data completeness in scenarios that require complex sets of views.

As a more detailed example, let $f_{\min} := 0.03$ cm be the smallest relevant feature of an examined object with maximal radius $r := 1cm$ and $k := 10$ be the magnification (due to cone-beam CT). Using the presented equations, we can calculate the maximal detector pixel size $\triangle\xi$ and the maximal angular gap $\triangle\gamma$ for data complete reconstruction:

$$\triangle\xi < \frac{k}{2} f_{\min} = 0.15 \text{ cm} \tag{8}$$

$$\triangle\gamma < \frac{f_{\min}}{2r} = 0.015. \tag{9}$$

This means, a set of projections provides complete data if, first, the pixel size is smaller than 0.15 cm and, second, for all positions $\mathbf{x}$ in the region of interest and possible vectors $\mathbf{u}$ through the region of interest, there exists a projection with a normal vector $d_{x,l}$ so that the maximal angular gap $\gamma_{\max} = \mathbf{d}_{x,l}^{\top}\mathbf{u} \leq \sin(0.015) \approx 0.015$.

For the following examples, we chose a sufficiently small pixel size of 0.12 cm and tested different sets of views. In the first example, we chose to reconstruct a spherical object with concentric spherical shells in order to visualise cone-beam and aliasing artefacts. Fig. 3, 4 and 5 show examples of different sets of views and slices of the corresponding reconstructions. With each a maximal angular gap of $\gamma_{\max} = 0.308$ at the bottom and the top of the sphere, both circular trajectories of Fig. 3 and 4 do not fulfil the presented data completeness conditions. Cone-beam artefacts appear at the bottom and the top due to a lacking sampling. Aliasing appears in Fig. 3 due to too few projections. Fig. 5 shows a set of views with 300 arbitrary projections that does fulfil the presented condition with a maximal angular gap of $\gamma_{\max} = 0.013$. It contains no aliasing or cone-beam artefacts as the corresponding Radon space is sampled sufficiently.

5. CONCLUSION

We have presented a CT data completeness condition that can be used to assess the completeness of any set of projections. This condition is not based on continuous curves, but can be applied directly to assess the completeness of data for any arbitrary set of projections. Thereby, this work introduces a method for evaluating sets of projections even for complex scanning scenarios, e.g. for robotic CT systems and scenarios with strongly attenuating components and spatial restrictions.

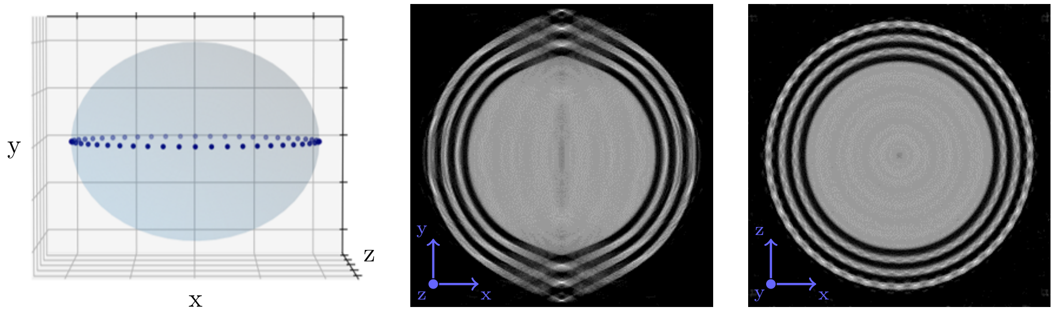

Figure 3. Circular trajectory with 50 projections: Visualisation of the source positions and two reconstruction slices of a spherical object. This set of projections is not data complete due to missing information outside of the centre plane and too few projections.

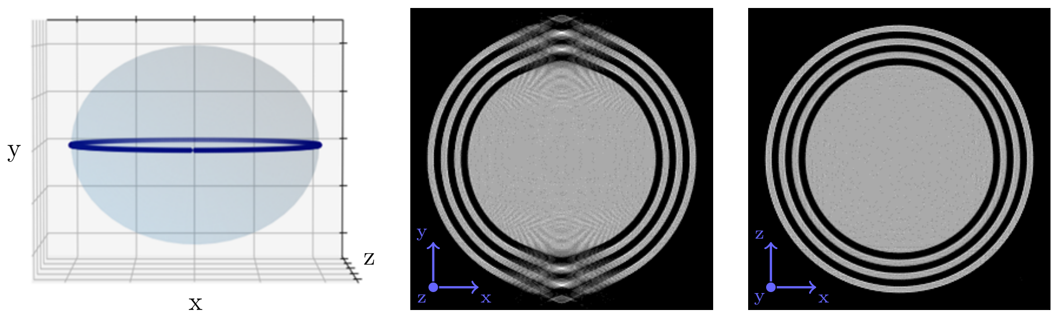

Figure 4. Circular trajectory with 300 projections: Visualisation of the source positions and two reconstruction slices of a spherical object. This set of projections is not data complete due to missing information outside of the centre plane.

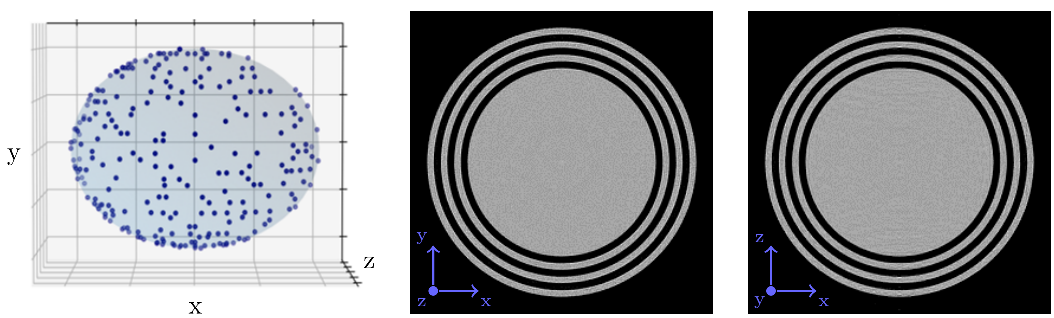

Figure 5. Data complete set of 300 arbitrary projections: Visualisation of the source positions and two reconstruction slices of a spherical object. This set of projections is data complete.

REFERENCES

[1] Rubin, G. D., "Computed tomography: revolutionizing the practice of medicine for 40 years," *Radiology* **273**(2S), S45–S74 (2014).

[2] Zabler, S., Maisl, M., Hornberger, P., Hiller, J., Fella, C., and Hanke, R., "X-ray imaging and computed tomography for engineering applications," *tm-Technisches Messen* **88**(4), 211–226 (2021).

[3] Holub, W., Brunner, F., and Schön, T., "Roboct-application for in-situ inspection of join technologies of large scale objects," in [*International Symposium on Digital Industrial Radiology and Computed Tomography*], (2019).

[4] Herl, G., Hiller, J., and Maier, A., "Scanning trajectory optimisation using a quantitative tuybased local quality estimation for robot-based x-ray computed tomography," *Nondestructive Testing and Evaluation* **35**(3), 287–303 (2020).

[5] Herl, G., Hiller, J., Thies, M., Zaech, J.-N., Unberath, M., and Maier, A., "Task-specific trajectory optimisation for twin-robotic x-ray tomography," *IEEE Transactions on Computational Imaging* (2021).

[6] Bauer, F., Goldammer, M., and Grosse, C. U., "Scan time reduction by fewer projections-an approach for part-specific acquisition trajectories," in [*20th World Conference on Non-Destructive Testing*], (2020).

[7] Smith, B. D., "Image reconstruction from cone-beam projections: necessary and sufficient conditions and reconstruction methods," *IEEE transactions on medical imaging* **4**(1), 14–25 (1985).

[8] Radon, J., "Über die bestimmung von funktionen längs gewisser mannigfaltigkeiten. sächsische gesellschaft der wissenschaften math," *Phys. Klasse, Leipzig* **69**, 262–277 (1917).

[9] Tuy, H. K., "An inversion formula for cone-beam reconstruction," *SIAM Journal on Applied Mathematics* **43**(3), 546–552 (1983).

[10] Buzug, T. M., [*Einführung in die Computertomographie: mathematisch-physikalische Grundlagen der Bildrekonstruktion*], Springer-Verlag (2003).

[11] Maier, A., Kugler, P., Lauritsch, G., and Hornegger, J., "Discrete estimation of data completeness for 3d scan trajectories with detector offset," in [*Bildverarbeitung für die Medizin 2015*], 47–52, Springer (2015).