Chapter 2

Imaging of self-luminous objects in terms of wave theory

§7. Diffraction problems solved on the basis of Maxwell's theory
We have seen that a centered system (microscope objective) images a surface element point-to-point and in similarity, using arbitrarily wide-angled ray bundles, only if the sine condition

$$\frac{\sin u'}{\sin u} = \frac{n}{n'} \cdot \frac{1}{\beta}$$

is fulfilled. If the system is so designed that this condition is satisfied, then all incoming rays to any point of the image remain perpendicular to a spherical surface centered on this point.[xvi] The lens designer[xvii] cannot offer anything more than this. We wonder whether and under what conditions this purely geometrical, pointwise concentration of rays is also physically present. Let us for the moment remain on the fiction of geometrical optics, that there were actually luminous points, so only the spherical wave emanating from this point would be a reality. Only with free, absolutely unhindered propagation, as would be the case in an arbitrarily extended, homogeneous medium, will the energy propagate along the radii exactly, as the ray theory assumes. If, however, as is always the case in reality, obstacles of any kind stand in the way of light propagation, i.e., if the medium exhibits inhomogeneities abruptly, light propagation can no longer be covered by ray-theoretic calculations; the wave fronts are no longer concentric spheres, but are somewhat deformed in a way (diffraction). The actually occurring propagation and distribution of the energy has been calculated based on Maxwell's electromagnetic theory of light only for very special cases.

The diffraction phenomenon appearing at the straight edge of an otherwise infinitely extended screen was treated by Sommerfeld.[1] Schwarzschild[2] succeeded in calculating the diffraction phenomenon associated with an infinitely extended slit of arbitrary width. Naturally, the numerical calculation becomes more difficult the smaller the slit width is in comparison to the wavelength. In addition, it must be emphasized that in both cases the material of the screen had to be assumed to have infinite conductivity. Under the same restriction, J. J. Thomson[3] could calculate the diffraction phenomenon of a sphere, whereas G. Mie[4] and P. Debye[5] carried out this case for spheres of arbitrary material. W. Seitz[6] and W. v. Ignatowsky[7] calculated the diffraction phenomenon of an infinitely long metallic cylinder of circular cross section and arbitrary conductivity, whereas Cl. Schaefer[8] carried out this calculation on cylinders of dielectric material and had it confirmed experimentally with the help of elec-

[1] *Mathem. Ann.* **47**, 317 (1896).
[2] ibid. **55** 177 (1902).
[3] J. J. Thomson, *Recent Researches in Electricity and Magnetism*, p. 361.
[4] *Ann. d. Phys.* **25**, 377 (1908).
[5] P. Debye, Dissertation. Munich 1908.
[6] *Ann. d. Phys.* **16**, 746 (1905); **19**, 554 (1906).
[7] *Ann. d. Phys.* **18**, 495 (1905).
[8] *Phys. Zeitschr.* X, **8**, 261.

trical waves (Großmann[9]). Finally, the diffraction phenomenon on metallic cylinders of elliptical cross section was treated (B. Sieger[10] and K. Aichi[11]), if only for material of infinitely large conductivity.

§8. The Kirchhoff principle

In general, the treatment of diffraction phenomena according to the Kirchhoff principle gives a far simpler form, allowing then the calculation of cases of our interest. Applying Green's theorems[xviii] to a function φ, which satisfies the wave equation[xix]

$$\frac{\partial^2 \varphi}{\partial t^2} = a^2 \Delta \varphi \,, \tag{12}$$

Kirchhoff[12] obtained the value of the function φ at an observation point P (Fig. 11) as a function of time t in terms of values of φ, $\partial\varphi/\partial t$, and $\partial\varphi/\partial\nu$ on the observation point–enclosing surface Σ with inward normal ν; here one must, for the magnitudes of φ, $\partial\varphi/\partial t$, and $\partial\varphi/\partial\nu$, insert the values that they possess at position $d\sigma$ at time $t' = t - r/a$, where r denotes the radius vector $P\,d\sigma$ and a the velocity of light in space V. It is[xx]

$$\varphi_P(t) = \frac{1}{4\pi} \int_\Sigma d\sigma \left[\varphi \frac{\partial(1/r)}{\partial\nu} - \frac{1}{ar}\frac{\partial\varphi}{\partial t} \cdot \frac{\partial r}{\partial\nu} - \frac{1}{r}\frac{\partial\varphi}{\partial\nu} \right]_{t'=t-\frac{r}{a}} . \tag{13}$$

Kirchhoff used this theorem to derive an approximation of the light intensity at observation point P (Fig. 12), if waves originating from L are disturbed by some obstacles. We want to carry out the calculation for the special case of an obstacle that is an *opaque* screen with aperture Σ_1. For this we place the surface of integration around

[9]Dissertation, Breslau 1909.

[10]*Ann. d. Phys.* **23**, 626 (1908).

[11]*Proc. Tokyo Mathem. Physical Soc.* (2) **4**, 966 (1908).

[12]Kirchhoff, *Lectures on Mathematical Physics, Vol. II, Optics*, 1891 (in German).

Figure 11

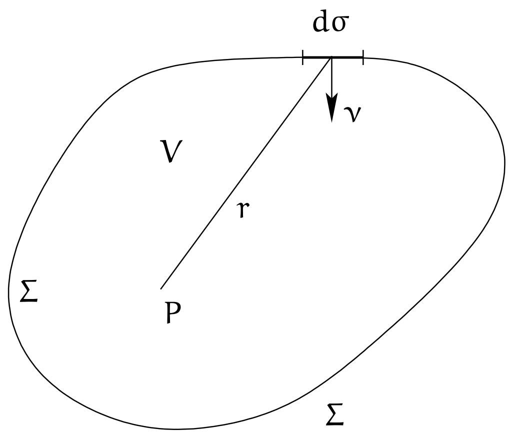
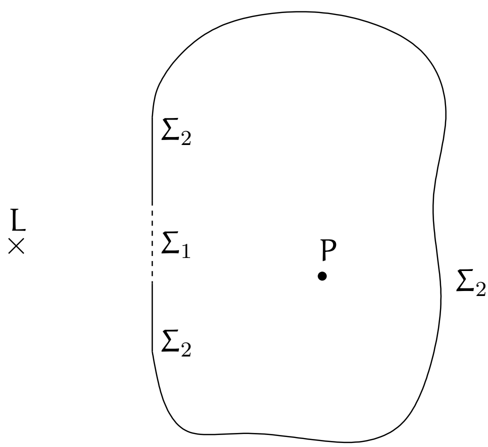


Figure 12

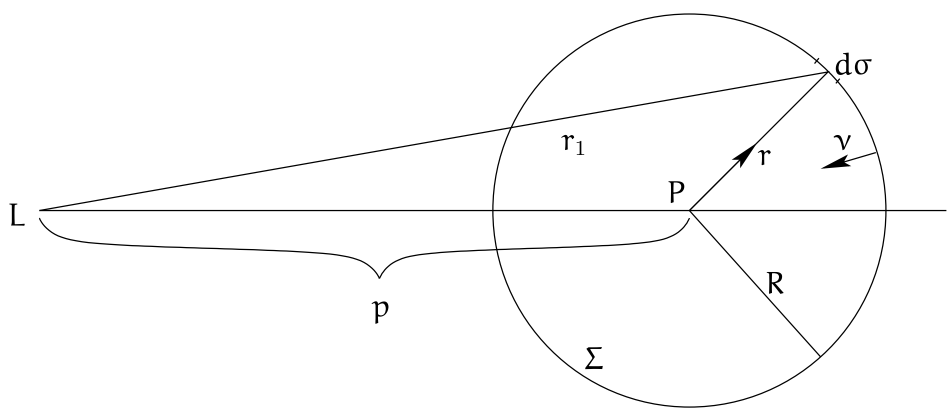


point P so that it is completely separated from L and let this surface consist of two parts, Σ_1 and Σ_2. Part Σ_2 wraps itself around the side of the screen facing the observation point and is thought of as closed at infinity. Let part Σ_1 be bordered by edges of the aperture.

The calculation of $\varphi_P(t)$ can only be carried out if one knows the values of φ, $\partial\varphi/\partial t$, and $\partial\varphi/\partial\nu$ at all points of the surface of integration; if one makes the natural hypothesis, *that the values on surface* Σ_1 *are the same as those of the undisturbed propagation*, and are zero on all points of surface Σ_2, then this assumption corresponds to the empirical knowledge that the bigger the aperture relative to the wavelength of the light, the closer it comes to the truth. In this case, the integral extends only over surface Σ_1.

The hypotheses made are strictly satisfied only for the *undisturbed* propagation. Here one knows the values of φ at P. We want to show that the calculation of φ by means of the Kirchhoff principle leads to this known value. For this we choose a sphere of radius R centered on P (Fig. 13) as the surface of integration and set, for points on surface Σ, as

Figure 13



$$\varphi = \frac{A}{r_1} \cos 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda} \right) .$$

Then we get

$$\frac{\partial \varphi}{\partial t} = -\frac{A}{r_1} \frac{2\pi}{T} \sin 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda} \right) ,$$

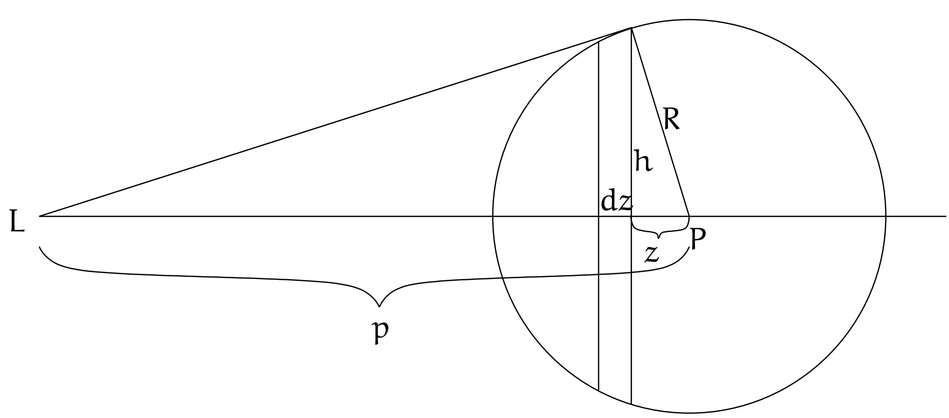
$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \varphi}{\partial r_1} \cos(r_1, \nu) = \cos(r_1, \nu) \left\{ -\frac{A}{r_1^2} \cos 2\pi \left(\frac{t}{T} - \frac{r_1}{\lambda} \right) + \frac{A}{r_1} \frac{2\pi}{\lambda} \cdot \sin 2\pi \left(\frac{t}{t} - \frac{r_1}{\lambda} \right) \right\} ,$$

$$\frac{\partial (1/r)}{\partial \nu} = -\frac{1}{r^2} \cos(r, \nu) = +\frac{1}{r^2} ,$$

$$\cos(r_1, \nu) = \frac{p^2 - r^2 - r_1^2}{2 r r_1} .$$

We take, as element $d\sigma$ (Fig. 14) of the surface of integration, the piece of surface that is sliced from the spherical surface by two planes

Figure 14



perpendicular to PL and separated from each other by a distance dz. We then have

$$d\sigma = 2\pi R\, dz \ .$$

Since according to Fig. 14 we have

$$\begin{aligned} r_1^2 &= (p-z)^2 + h^2 \\ R^2 &= z^2 + h^2 \ , \end{aligned}$$

it follows then

$$r_1^2 = R^2 + p^2 - 2pz \ .$$

Differentiating this equation gives

$$dz = -\frac{r_1\, dr_1}{p} \ ,$$

where the limits of integration with respect to r_1 are $p-R$ and $p+R$. Inserting all these values, we have

$$\begin{aligned} \varphi_P(t) = -\frac{1}{4\pi} \int_{p+R}^{p-R} \frac{2\pi R r_1\, dr_1}{p} \Bigg\{ \frac{A}{r_1 R^2} \cos\vartheta - \frac{A 2\pi}{a R r_1 T} \sin\vartheta \\ - \frac{A(p^2 - R^2 - r_1^2)}{R r_1 \cdot 2 R r_1} \left(-\frac{\cos\vartheta}{r_1} + \frac{2\pi}{\lambda} \sin\vartheta \right) \Bigg\} \ , \end{aligned}$$

where

$$\vartheta = 2\pi \left(\frac{t}{T} - \frac{R + r_1}{\lambda} \right) \ ;$$

or recast,

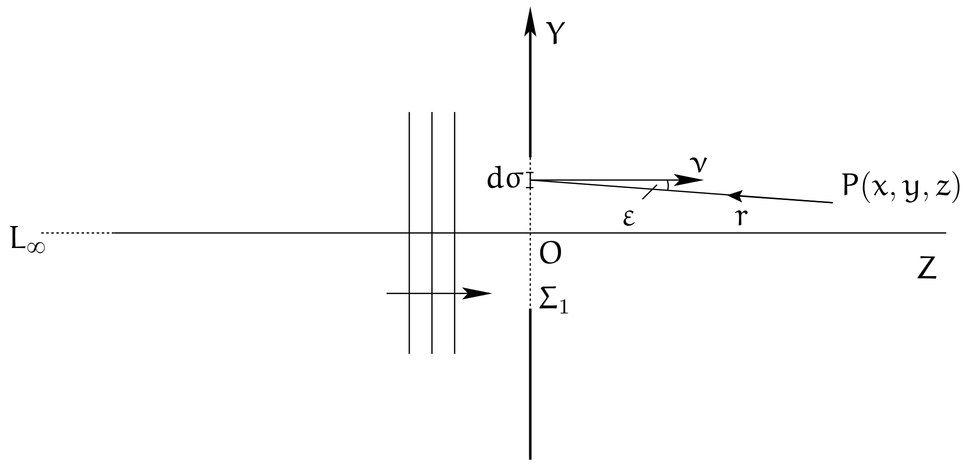
$$\begin{aligned}\varphi_P(t) &= -\frac{A}{2p}\int_{p+R}^{p-R} dr_1 \left\{ \frac{\cos\vartheta}{R} - \frac{2\pi\sin\vartheta}{\lambda} + \frac{p^2-R^2-r_1^2}{2Rr_1} \right. \\ &\qquad \left. \times \left(\frac{\cos\vartheta}{r_1} - \frac{2\pi}{\lambda}\sin\vartheta\right)\right\} \\ &= -\frac{A}{2p}\int_{p+R}^{p-R} dr_1 \left\{ \frac{\cos\vartheta}{R}\left(1+\frac{p^2-R^2-r_1^2}{2r_1^2}\right) \right. \\ &\qquad \left. -\frac{2\pi\sin\vartheta}{\lambda}\left(1+\frac{p^2-R^2-r_1^2}{2Rr_1}\right)\right\} \\ &= +\frac{AR}{2p}\int_{p+R}^{p-R} dr_1 \frac{d\left[\frac{\cos\vartheta}{R}\left(1+\frac{p^2-R^2-r_1^2}{2r_1R}\right)\right]}{dr_1} \\ &= \frac{AR}{2p}\left[\frac{\cos\vartheta}{R}\left(1+\frac{p^2-R^2-r_1^2}{2r_1R}\right)\right]_{p+R}^{p-R} = \frac{A}{p}\cos 2\pi\left(\frac{t}{T}-\frac{p}{\lambda}\right) ,\end{aligned}$$

i.e., the light disturbance taking place at P for the undisturbed propagation.

We now want to calculate the diffraction phenomenon caused by an arbitrary aperture in a *planar* screen for the case in which the point of light L is situated infinitely far from the diffraction aperture, that is, a *plane* wave is perpendicularly incident on the screen. The xy-plane (Fig. 15) is to lie in the plane of the screen, and the piece let go from the screen (diffracting aperture) is chosen as the surface of integration Σ_1. As the expression of the light disturbance φ, we set

$$\varphi = A\cos 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right) .$$

Figure 15



We then have

$$\begin{aligned}
\frac{1}{a}\frac{\partial\varphi}{\partial t} &= -\frac{2\pi A}{\lambda}\sin 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right) \,, \\
\frac{\partial\varphi}{\partial\nu} &= \frac{\partial\varphi}{\partial z} = \frac{2\pi A}{\lambda}\sin 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right) \,, \\
\frac{\partial r}{\partial\nu} &= \cos(r,z) = -\cos\varepsilon \,,
\end{aligned}$$

and therefore

$$\varphi_P(t) = \frac{1}{4\pi}\int d\sigma\left\{\frac{A}{r^2}\cos\varepsilon\cos\vartheta - \frac{2\pi A\cos\varepsilon}{r\lambda}\sin\vartheta - \frac{2\pi A}{r\lambda}\sin\vartheta\right\} \,,$$

where

$$\vartheta = 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) \,.$$

If the distance r from the aperture to point P is large compared to the wavelength λ, the first term in the braces is negligible compared to the other two terms, and we obtain

$$\varphi_P(t) = -\frac{A}{\lambda}\int \frac{d\sigma}{r}\frac{1+\cos\varepsilon}{2}\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) . \tag{14}$$

§9. Discussion of expression for the intensity at the observation point

From here, if one forms the average value $\overline{\varphi_P^2(t)}$,[xxi] it is then a direct measure of the observed intensity at observation point P; this is a consequence of the fact that we have used the ansatz of φ being a *plane* wave. For clarification, we note the following: according to the electromagnetic theory of light, the intensity of the field at every position is given by $\overline{\mathfrak{E}^2}$, where $\mathfrak{E}$ is simply the electric vector at the place of observation. For illustration, the following useful solution of Maxwell's equations is well known for spherical waves as well as for plane waves:[xxii]

$$\mathfrak{E}_x = \frac{\partial^2\varphi}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} , \qquad \mathfrak{H}_x = 0 ,$$

$$\mathfrak{E}_y = \frac{\partial^2\varphi}{\partial y\partial x} , \qquad \mathfrak{H}_y = +\frac{1}{a}\frac{\partial^2\varphi}{\partial z\partial t} ,$$

$$\mathfrak{E}_z = \frac{\partial^2\varphi}{\partial z\partial x} , \qquad \mathfrak{H}_z = -\frac{1}{a}\frac{\partial^2\varphi}{\partial y\partial t} ,$$

where φ must satisfy the equation[xxiii]

$$\frac{\partial^2\varphi}{\partial t^2} = a^2\Delta\varphi .$$

Here, $\mathfrak{E}$ and $\mathfrak{H}$ designate electric and magnetic vectors of the field. Let us start with a plane wave

$$\varphi = A\cos 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right) ,$$

and get

$$\mathfrak{E}_x = \frac{4\pi^2 A}{\lambda^2}\cos 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right), \qquad \mathfrak{H}_x = 0,$$
$$\mathfrak{E}_y = 0, \qquad \mathfrak{H}_y = \frac{4\pi^2 A}{\lambda^2}\cos 2\pi\left(\frac{t}{T}-\frac{z}{\lambda}\right),$$
$$\mathfrak{E}_z = 0, \qquad \mathfrak{H}_z = 0.$$

Therefore,

$$\overline{\mathfrak{E}^2} = \overline{\mathfrak{E}_x^2} = \frac{1}{2}\left(\frac{4\pi^2 A}{\lambda^2}\right)^2 = \frac{8\pi^4}{\lambda^4}A^2.$$

On the other hand, $\overline{\varphi^2} = \frac{1}{2}A^2$, which illustrates that, in the case of *plane* waves, $\overline{\varphi^2}$ differs from $\overline{\mathfrak{E}^2}$, which is relevant for the intensity, by only a constant factor, and that $\overline{\varphi^2}$ may be seen as a measure of the intensity.

The case of *spherical waves* is different, for which we have to start with[xxiv]

$$\varphi = \frac{A}{r}\cos 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right),$$

where $r = \sqrt{x^2+y^2+z^2}$. If r is large compared to λ, then we get

$$|\mathfrak{E}| = \frac{4\pi^2 A\sin\vartheta}{\lambda^2 r}\cos 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right), \tag{15}$$

where ϑ is the angle formed by the radial vector r with the x-axis.[xxv] From this it follows then

$$\overline{\mathfrak{E}^2} = \frac{8\pi^4 A^2\sin^2\vartheta}{\lambda^4 r^2},$$

whereas

$$\overline{\varphi^2} = \frac{A^2}{2r^2};$$

one therefore sees that, with spherical waves, one may not regard the last expression of $\overline{\varphi^2}$ as a measure of the intensity, because the true intensity $\overline{\mathfrak{E}^2}$ still varies with the direction ϑ at a constant r.

We may add that the field determined by Eq. 15 can be viewed as originating from an electric dipole or Hertzian oscillator whose axis of oscillation coincides with the x-axis.

In reality, one deals with the radiation of spatial objects that can be thought of as filled with radiating dipoles. In order to give a concept of the number of such dipoles, we must know the ratio of the number of radiating to the number of overall available molecules per unit volume. If we take luminous hydrogen as a basis and make the assumption that every molecule possesses one electron, then in every cubic centimeter, according to Ladenburg-Loria,[13] only 4×10^{12} are so-called radiating "dispersion electrons," compared to 2×10^{17} overall available electrons (molecules). In a cube of luminous hydrogen with an edge length of 0.001 mm = 1 μm, there would then still be about four dispersion electrons present. In luminous vapors, however, even more dispersion electrons are present in such a volume element; in sodium vapor, e.g., there are about 1000. In reality, in radiating gases or vapors, we are not even dealing with individual undisturbed oscillating dipoles. On the other hand, we know that in radiating black bodies every surface element radiates according to Lambert's cosine law,[14] so that in free radiation the intensity at observation point P (Fig. 16) has the value

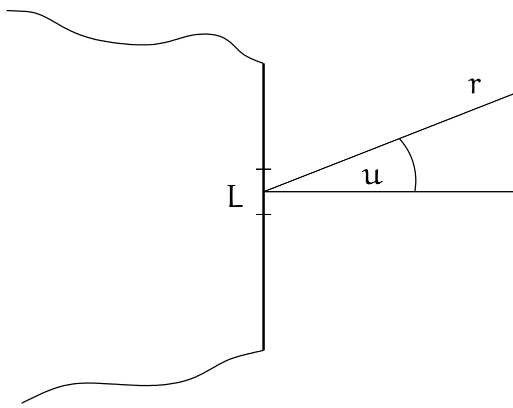
$$\frac{A^2}{r^2} \cos u \,;$$

here, too, the intensity depends on the direction of radiation r. Therefore, $\overline{\varphi^2}$ is a measure of intensity in neither free nor disturbed light propagation. Only when the luminous surface element is situated

[13] *Phys. Zeitschr.* (9) **24**, 875.

[14] O. Lummer and F. Reiche, *Dependence of radiation from a "Bunsen plate" (Bec Méker) on the radiating angle,* Verh. d. Schles. Ges. f. V. K. (1910) (in German).

Figure 16



so far from the diffracting aperture that we can consider the incident waves as planar, may we regard $\overline{\varphi_P^2(t)}$ as a measure of intensity.

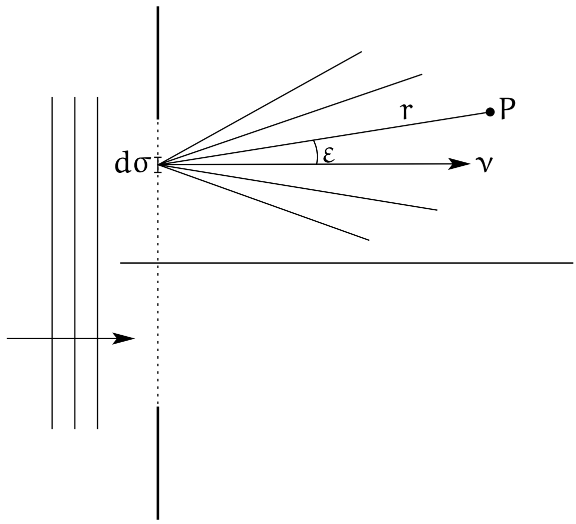
§10. Comparison of the Kirchhoff principle with the Fresnel–Huygens principle

We return to our expression (Eq. 14) for the light disturbance occurring at observation point P behind the diffraction aperture. It is

$$\varphi_P(t) = -\frac{A}{\lambda}\int \frac{d\sigma}{r}\frac{1+\cos\varepsilon}{2}\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) .$$

In this version, we can interpret our formula as an expression of the Fresnel–Huygens principle, according to which one obtains the resulting light disturbance at observation point P due to the interference of imaginary coherent elemental waves leaving from all elements of the diffraction aperture. In our experience, the formula leading to correct results shows which factors to use when one takes into account the contribution of individual elemental waves; we can

Figure 17



write the contribution of each surface element dσ (Fig. 17) of the diffracting aperture as

$$-\frac{A'}{r}\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right)=\frac{A'}{r}\cos\left[2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right)+\frac{\pi}{2}\right],$$

where

$$A'=\frac{A\,d\sigma}{\lambda}\left(\frac{1+\cos\varepsilon}{2}\right).$$

Therefore, it is as if every element dσ sends out a spherical wave whose amplitude is A' at the unit distance, and whose phase with respect to that of the incident wave has been shifted by $\pi/2$. The amplitude, which one must enclose in the elemental waves in the direction of r, is to be set proportional to $\frac{1+\cos\varepsilon}{2}$, where ε is the angle between r and the incident direction of the impinging radiation.

In other words, it means that every surface element $d\sigma$ should not radiate according to Lambert's cosine law, according to which the amplitude would be proportional to $\sqrt{\cos\varepsilon}$; instead, the amplitude should vary proportional to $\frac{1+\cos\varepsilon}{2}$. One may easily see that both laws agree with each other up to the terms of order ε^2.[xxvi] However, there is absolutely no reason for assuming that these so defined elemental waves represent any kind of reality.

Fresnel made qualitatively similar assumptions in order to calculate the diffraction effect of an aperture. According to him, different surface elements contribute to the light disturbance at the observation point (1) proportional to its size; (2) inversely proportional to the distance from the observation point; and (3) proportional to a factor dependent on the direction with respect to the normal, with the normal direction being the maximum. Except for the phase of the oscillation, the Fresnel–Huygens principle also describes correctly the intensity distribution at least at a relatively large distance from the diffraction screen.

§11. Fraunhofer diffraction

One becomes independent of this proportionality factor, which is $\left(\frac{1+\cos\varepsilon}{2}\right)$ according to the Kirchhoff principle, if one lets the observation point go to *infinity*. To find the form that the phase takes in this case, we start from the relationship

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2 ,$$

where x, y, z are the coordinates of the observation point and ξ, η, 0 are those of element $d\sigma$. If we set

$$x^2 + y^2 + z^2 = r_0^2 ,$$

it follows then

$$r = r_0\sqrt{1 + \frac{\xi^2 + \eta^2 - 2(x\xi + y\eta)}{r_0^2}} ;$$

if we let r_0 grow without restraint, ξ and η will always be small compared to r_0 and one can expand the square root in the following manner:[xxvii]

$$r = r_0 \left\{ 1 + \frac{\xi^2 + \eta^2}{2r_0^2} - \frac{x\xi + y\eta}{r_0^2} - \frac{(x\xi + y\eta)^2}{2r_0^4} \right\} .$$

If we set for the moment $x/r_0 = \alpha$, $y/r_0 = \beta$, we get

$$r = r_0 - \left((\xi\alpha + \eta\beta) + \frac{\xi^2 + \eta^2 - (\xi\alpha + \eta\beta)^2}{2r_0} \right) .$$

And so for infinitely large r_0,

$$r = r_0 - (\xi\alpha + \eta\beta) = r_0 - \frac{x\xi + y\eta}{r_0} .$$

Therefore,

$$\varphi_P(t) = -\frac{A}{\lambda} \int \frac{d\sigma}{r} \sin 2\pi \left(\frac{t'}{T} + \frac{x\xi + y\eta}{r_0 \lambda} \right) , \qquad (16)$$

where we set $t' = t - r_0/c$.

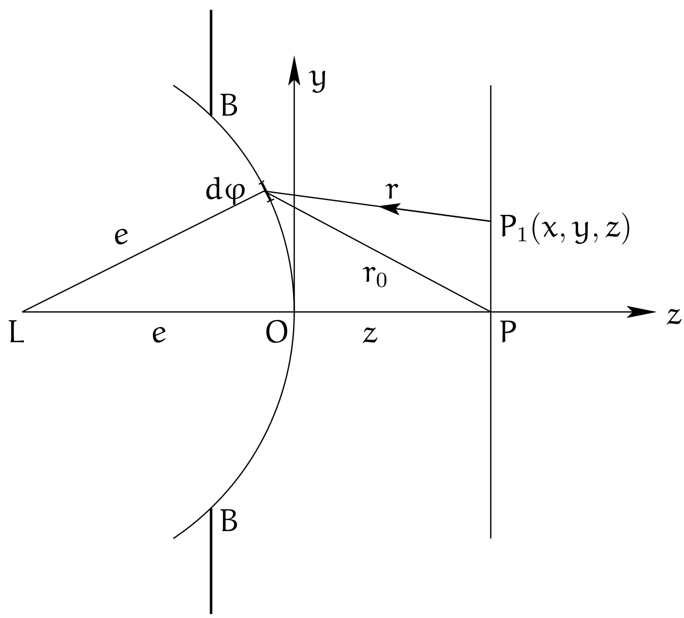
The phenomenon given by this expression is called Fraunhofer diffraction; it is exceptional in both formal and physical respects. Whereas with *finite* distance, be it of the luminous point or the observation point (Fresnel diffraction), *quadratic* terms in ξ and η appear in the expression for the phase, they *disappear* in Fraunhofer diffraction in which *the luminous point and the observation point lie at infinity*. This is realized if one brings the luminous point to the focal plane of a convex lens and observes the phenomenon in the focal plane of a second convex lens. Light source and observation point therefore lie in the planes that are, with respect to the imaging system (the two convex lenses), *conjugate* to each other. We want to show that we always get Fraunhofer diffraction; i.e., we always retain only *linear* terms in ξ and η in the expression for the phase if we make the luminous point and the observation point an *arbitrary conjugate* pair of points with respect to the imaging system. For this we investigate an auxiliary consideration.

§12. Auxiliary consideration

Let the diaphragm BB (Fig. 18) cut out, from the spherical wave coming from L, a piece of surface BOB that we choose as the surface of integration. If $d\varphi$ is an element of that surface and r is the distance between this element and observation point P_1, then we can depict the light disturbance at P_1 using the expression

$$s = \int \frac{A}{e}\frac{\mu}{\lambda}\frac{1}{r}\, d\varphi \sin 2\pi \left(\frac{t}{T} - \frac{e+r}{\lambda}\right) ,$$

Figure 18



where A/e is the amplitude of the light disturbance at $d\varphi$ and the factor μ takes into account the inclination of the elemental ray r with respect to $d\varphi$.

We choose O as the origin of a rectangular Cartesian coordinate system, LOP as the z-axis, the line through O pointing upward and perpendicular to LOP as the y-axis, and the line perpendicular to the drawing going into the paper as the x-axis.

If $\xi\eta\zeta$ are the coordinates of $d\varphi$, xyz are those of P_1, and we designate line segment $P\,d\varphi$ as r_0, then

$$\begin{aligned} r^2 &= (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \\ &= (x^2+y^2) - 2(x\xi + y\eta) + r_0^2 \,. \end{aligned}$$

The equation of the sphere is valid for the coordinates of $d\varphi$:

$$\xi^2 + \eta^2 + (e+\zeta)^2 = e^2 \text{ or } \xi^2 + \eta^2 = -\zeta^2 - 2e\zeta \,.$$

Therefore,

$$r_0^2 = \xi^2 + \eta^2 + (z-\zeta)^2 = (z-\zeta)^2 - \zeta^2 - 2e\zeta = z^2 - 2\zeta(z+e) \,.$$

r_0^2 takes on a particularly simple value if

$$z = -e \,.$$

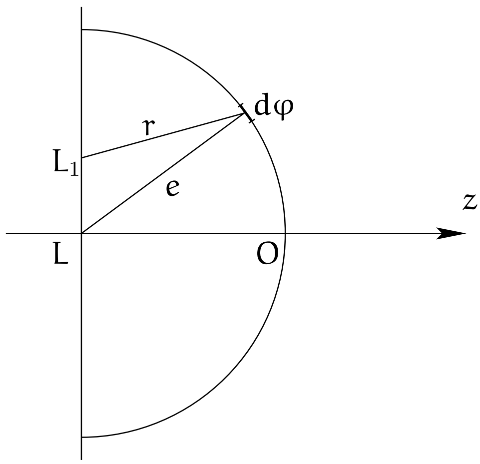
Then (Fig. 19),

$$r_0^2 = e^2 \text{ and } r^2 - r_0^2 = (r+e)(r-e) = x^2 + y^2 - 2(x\xi + y\eta) \,.$$

If we set

$$r - e = \rho \text{ and therefore } r + e = \rho + 2e \,,$$

Figure 19



then the following equation is valid:

$$\rho^2 + 2e\rho + [2(x\xi + y\eta) - (x^2 + y^2)] = 0 \,.$$

It follows that

$$\rho = -e + \sqrt{e^2 - [2(x\xi + y\eta) - (x^2 + y^2)]}$$

or

$$\rho = -e + e\sqrt{1 - 2\frac{x\xi + y\eta - \frac{x^2+y^2}{2}}{e^2}} \,.$$

If x and y are small compared to e, i.e., if one limits oneself to observation points close to the line LOP, then

$$\rho = -e + e\left(1 - \frac{x\xi + y\eta}{e^2}\right) \,,$$

or finally,

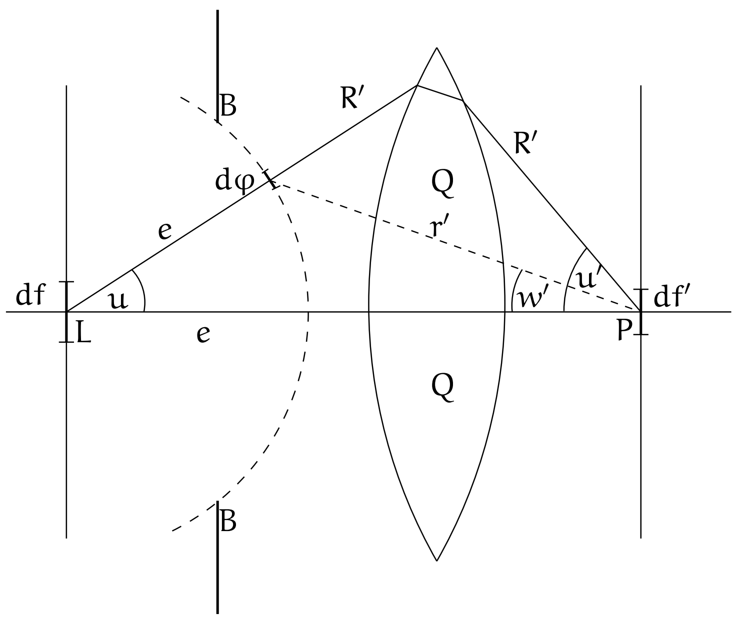
$$\rho = -\frac{x\xi + y\eta}{e} . \tag{17}$$

This simplification of the value ρ for $z = -e$, i.e., for the observation points *that lie in the object plane itself*, acquires a *physical* meaning with the introduction of imaging systems.

§13. Diffraction phenomena occurring in pairs of conjugate planes of optical systems

In Fig. 20, let the surface element df lying at L glow and its image df', projected by system Q, lie at P. Let diaphragm BB act as the entrance pupil that cuts an effective piece of the surface out of a

Figure 20



sphere centered at L with radius e. Let $d\varphi$ be an element of the surface; then the "amplitude" of the outgoing wave from df at $d\varphi$ is $\frac{A}{e} = \alpha$, where the designation "amplitude" is so understood that the intensity at the location of $d\varphi$ is given by the expression

$$J_{d\varphi} = \alpha^2 \, df = \frac{A^2}{e^2} \, df \ .$$

If we designate the rectilinear distance (dotted) from $d\varphi$ to P as r', then according to the Huygens principle, the *without-the-lens* light disturbance at P due to $d\varphi$ would have the amplitude

$$\frac{1}{\lambda} \alpha \, d\varphi \, \frac{1}{r'} \psi(w') \ ,$$

where $\psi(w')$ should take into account, with the interference of elemental waves, the influence of the inclination of the various elemental rays r' with respect to the direction of the axis LP and the inclination of the element $d\varphi$ to the associated elemental ray r'.

In the presence of the lens, from each element $d\varphi$ come the elemental rays that run in the immediate vicinity of chief ray R' associated with $d\varphi$, where R' also denotes the path length from $d\varphi$ toward P. With the lens we can therefore set the amplitude of the light disturbance at P originating from $d\varphi$ as

$$\frac{1}{\lambda} \alpha \, d\varphi \, f(R') \psi(u') \ ,$$

where $\psi(u')$ takes into account the various inclinations of the interfering elemental waves with respect to the axis and $f(R')$ their various geometrical lengths. The inclination of $d\varphi$ with respect to the effective elemental waves going out from $d\varphi$ is the same for all $d\varphi$. Since the geometrical length R' depends only on the accompanying angle of divergence u,[xxviii] we can then set

$$f(R') = \sigma(u) \ ,$$

and the resulting disturbance at P becomes

$$s = \frac{1}{\lambda} \int \alpha \, d\varphi \, \sigma(u)\psi(u') \sin 2\pi \left(\frac{t}{T} - \delta_P \right) ,$$

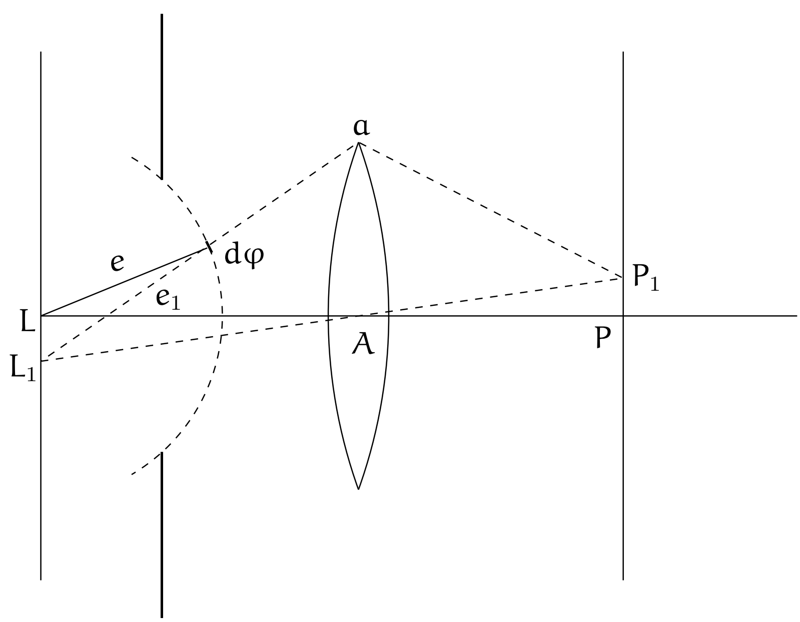
if δ_P designates the equal *optical* path length for all elemental rays between L and P.

The intensity at P is then given by

$$J_P = \overline{s^2} \, df .$$

Toward a point P_1 (Fig. 21) in the image plane come elemental pencils from $d\varphi$ that are seemingly coming from L_1, which is the

Figure 21



conjugate point of P_1. If the observation points are limited to be very close to the axis, one can express the resulting disturbance at P_1 as

$$s = \frac{1}{\lambda} \int \alpha \, d\varphi \, \sigma(u)\psi(u') \sin 2\pi \left(\frac{t}{T} - \delta_{P_1} \right) ,$$

where δ_{P_1} is the sum of the optical path length $\overline{Ld\varphi}$ and the optical path length $\overline{d\varphi a P_1}$. Now

$$\overline{d\varphi a P_1} = \overline{L_1 A P_1} - \overline{L_1 d\varphi} ,$$

where $\overline{L_1 A P_1}$ is a constant for the fixed location of P_1 and varies with the location of P_1.

Therefore,[xxix]

$$\overline{L d\varphi P_1} = \text{const} - (\overline{L_1 d\varphi} - \overline{L d\varphi})$$

and with that

$$\delta_{P_1} = -\frac{e_1 - e}{\lambda} + \text{const} = -\frac{\rho}{\lambda} + \text{const},$$

if one designates the segment $L_1 \, d\varphi$ by e_1. If P_1 moves toward P, $e_1 = e$ and the above constant becomes equal to δ_P. The phase difference between P and P_1 is exactly the same as that between their conjugate points L and L_1. If we designate the coordinates of L_1 by x, y, z and those of $d\varphi$ by ξ, η, ζ, then, as described earlier,[xxx]

$$\delta_{P_1} = \text{const} - \frac{\rho}{\lambda} = \text{const} + \frac{y\eta + x\xi}{e\lambda} .$$

With this, we obtain the resulting disturbance at P_1:

$$s = \frac{1}{\lambda} \int \alpha \, d\varphi \, \sigma(u)\psi(u') \sin 2\pi \left(\frac{t}{T} - \frac{x\xi + y\eta}{e\lambda} \right) , \tag{18}$$

where the constant phase difference is lumped into t.

It should be pointed out here once and for all that in the expression for the light disturbance at observation point P_1 in the image plane found according to rules of geometrical optics, the coordinates of the observation point itself do not appear. Rather, the coordinates xy of the P_1-conjugate point L_1 in the object plane appear. Actually, we would have to substitute x and y with the expression

$$x = x'/\beta\,, \quad y = y'/\beta\,,$$

where $x'y'$ designate the coordinates of P_1 and β designates the lateral magnification. We do not, however, want to carry out this substitution because it only complicates the discussion of the expression of s and does not change the essence of the matter. The intensities calculated using pairs $x'y'$ and xy are exactly the same. If one depicts the diffraction phenomenon calculated in the image plane according to the rules of geometrical optics in the object plane, this depicted phenomenon is identical with the phenomenon calculated using the object points xy according to Eq. 18. One would see this phenomenon by replacing the optical system Q with the eye and accommodating on the object plane. In this respect, we are entitled to designate the phenomenon depicted by the expression s the "diffraction phenomenon in the object plane."

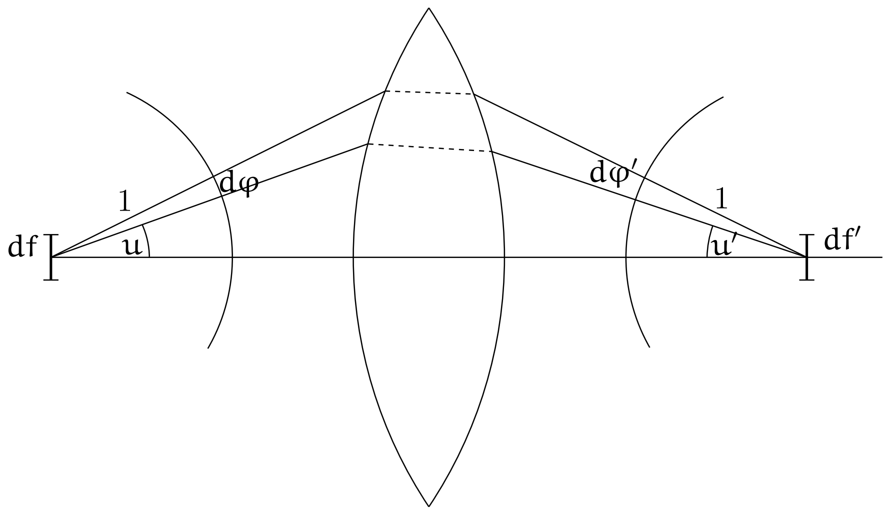
§14. Determination of factors α, $\sigma(u)$, and $\psi(u')$ based on energy considerations

To determine $\sigma(u)$, we presuppose that the sine condition is fulfilled. The energy principle says that in this case, the entire energy striking the system from object element df (Fig. 22) must flow through the point-to-point conjugate and similar image element df'. Since the same amount of energy must flow into conjugate elemental cones, we have

$$df \cdot d\varphi \cdot A^2 = df'\, d\varphi'\, A'^2\,,$$

if $d\varphi$ and $d\varphi'$ denote those surface elements that the elemental cones cut out of unit spheres about df and df', and A and A' denote

Figure 22



amplitudes present at $d\varphi$ and $d\varphi'$. If β denotes the lateral magnification of the system, then

$$\beta^2 = \frac{A^2\, d\varphi}{A'^2\, d\varphi'} \, .$$

If one introduces polar coordinates in a known manner,[xxxi] then

$$d\varphi = \sin u \, du \, dv$$
$$d\varphi' = \sin u' \, du' \, dv \, .$$

Therefore,

$$\frac{d\varphi}{d\varphi'} = \frac{\sin u \, du}{\sin u' \, du'} \, .$$

One obtains a relationship between u and u' using the sine condition[xxxii]

$$\sin u' = \frac{\lambda'}{\lambda} \cdot \frac{1}{\beta} \sin u \, ,$$

where β denotes lateral magnification. Differentiation of the above expression yields

$$\cos u'\, du' = \frac{\lambda'}{\lambda} \cdot \frac{1}{\beta} \cdot \cos u\, du$$

and therefore

$$d\varphi' = d\varphi \cdot \left(\frac{\lambda'}{\lambda}\right)^2 \cdot \left(\frac{1}{\beta}\right)^2 \frac{\cos u}{\cos u'} .$$

If one inserts this value of $d\varphi'$ into the energy equation, it follows then

$$\frac{A'^2}{A^2} = \frac{\lambda^2 \cos u'}{\lambda'^2 \cos u} = \frac{n'^2 \cos u'}{n^2 \cos u} . \tag{19}$$

If $u' = 0$, i.e., the image moves to infinity, then

$$A^2 = \frac{n^2 \cos u}{n'^2} \cdot A'^2 .$$

Only when A'^2 is a constant for all elemental cones, i.e., when *the plane wave front has the same intensity everywhere in the image space*, does the above relationship transition to the law

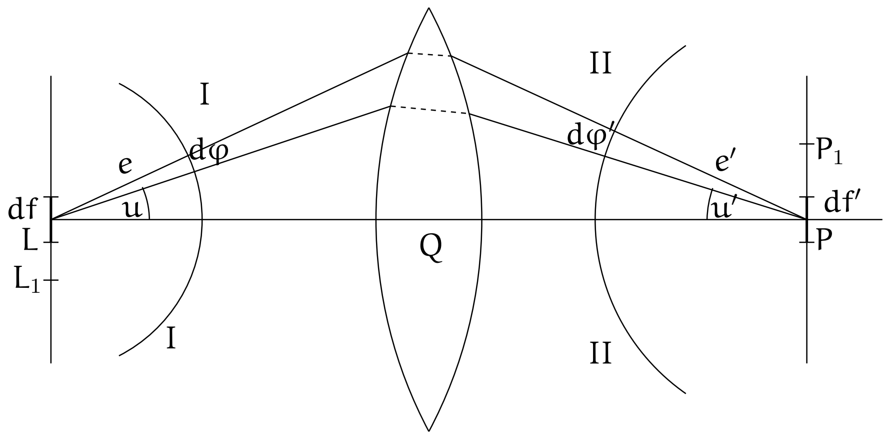
$$A^2 = \text{const}\, n^2 \cdot \cos u , \tag{20}$$

which represents the combination of the Lambert cosine law with the Kirchhoff–Clausius law of radiation.

We now construct the resulting light disturbance at P_1 while we consider, as boundary surfaces, one surface I situated at the distance e (Fig. 23) with elements $d\varphi$ and the other surface II located in the image space with elements $d\varphi'$. Let us denote the light disturbance at P_1 based on the boundary surface I as s_1; then, as before, we get

$$s_1 = \frac{1}{\lambda} \int\limits_{I} \alpha\, d\varphi\, \sigma(u)\psi(u') \sin 2\pi \left(\frac{t}{T} - \frac{x\xi + y\eta}{e\lambda}\right) ,$$

Figure 23



where x, y, $-e$ are the coordinates of P_1's conjugate point L_1, and ξ, η are the coordinates of $d\varphi$. On the basis of boundary surface II,

$$s_2 = \frac{1}{\lambda'} \int\limits_{II} \frac{\alpha'}{e'}\, d\varphi'\, \psi(u') \sin 2\pi \left(\frac{t}{T} + \frac{x'\xi' + y'\eta'}{e'\lambda'} \right) \;;$$

here, $\sigma(u)$ is replaced by $1/e'$ since our surface of integration, in the sense of light propagation, is located *after* the system Q; x', y' are the coordinates of P_1 and ξ', η' are those of $d\varphi'$.

If we introduce polar coordinates by making the substitution

$$\begin{aligned} \xi &= e \sin u \cos v \\ \eta &= e \sin u \sin v \\ d\varphi &= e^2 \sin u \, du \, dv \;, \end{aligned}$$

we get

$$s_1 = \frac{1}{\lambda}\int_0^{2\pi} dv \int_0^{U} du\, \alpha\sigma(u)\psi(u')e^2 \sin u \sin 2\pi\left(\frac{t}{T} - \sin u \frac{x\cos v + y \sin v}{\lambda}\right),$$

where U denotes the half angle of the aperture in the object space. Let us also introduce polar coordinates in s_2 and set in addition

$$x' = x\beta\,, \quad y' = y\beta\,.$$

If one bears in mind that for $\beta < 0$, ξ and ξ' as well as η and η' have the same sign, but x and x' as well as y and y' have opposite signs, whereas the reverse occurs for $\beta > 0$; by considering the sine condition,[xxxiii] one obtains

$$s_2 = \frac{1}{\lambda'}\int_0^{2\pi} dv \int_0^{U} du\, \alpha' e' \left(\frac{\lambda'}{\lambda}\right)^2 \left(\frac{1}{\beta}\right)^2 \frac{\cos u}{\cos u'} \psi(u') \sin u \cdot$$
$$\sin 2\pi\left(\frac{t}{T} - \sin u \frac{x\cos v + y \sin v}{\lambda}\right)\,.$$

By equating s_1 and s_2, we obtain the relation

$$\alpha' e' \frac{\lambda'}{\lambda}\left(\frac{1}{\beta}\right)^2 \frac{\cos u}{\cos u'} = \alpha e (e\sigma)$$

or

$$A' \frac{\lambda'}{\lambda} \cdot \frac{1}{\beta^2} \frac{\cos u}{\cos u'} = A e\sigma\,.$$

By using the value of $\frac{A'}{A}$ $\left(= \frac{\alpha' e'}{\alpha e}\right)$ obtained from the energy principle,[xxxiv] we finally obtain

$$e\sigma = \frac{1}{\beta^2}\sqrt{\frac{\cos u}{\cos u'}}\,. \tag{21}$$

To determine $\psi(u')$, we construct for one time the resulting light disturbance at P under the premise that L glows and using surface II as the boundary surface. For another time, the resulting light disturbance is at L under the premise that P, the image of L, glows and using surface I as the intermediate surface. One can think of realizing this assumption by setting up a perfect mirror perpendicular to the axis at the location of P. The resulting light disturbance is given by the expression s_2 in the first case, if one sets $x = y = 0$ in it; for the case that P, the image of L, glows and I is used as the boundary surface, we obtain, for the light disturbance at L,

$$s_1' = \frac{1}{\lambda}\int d\varphi\,\frac{\alpha}{e}\psi(u)\sin 2\pi\frac{t}{T}$$

or, in polar coordinates,

$$s_1' = \frac{1}{\lambda}\int\limits_0^{2\pi} d\nu \int\limits_0^{U} du\,\alpha e \sin u\,\psi(u)\sin 2\pi\frac{t}{T}\ .$$

The amplitudes of the light disturbance at P (if L glows) and at L (if P glows) follow a known relationship. To determine this relationship, let us consider the following.

The contribution that the element $d\varphi'$ provides to the light disturbance is

$$ds_2 = B'\sin 2\pi\frac{t}{T}\ ,$$

where

$$B' = \frac{1}{\lambda'}\,d\nu\,du\,\alpha' e'\left(\frac{\lambda'}{\lambda}\right)^2\frac{1}{\beta^2}\frac{\cos u}{\cos u'}\psi(u')\sin u\ .$$

We ask ourselves how large the resulting intensity caused by this contribution at P is. It is just as large as if df' itself radiated. That is,

$$J_P = \overline{ds_2^2}\,df' = \frac{1}{2}B'^2\,df'\ ,$$

and therefore the energy that flows through df' in time dt is

$$E_P = J_P\, df'\, dt = \frac{1}{2} B'^2 (df')^2\, dt \;.$$

Analogously, if df' radiates, the energy flowing through df that comes from $d\varphi$ is

$$E_L = \frac{1}{2} B^2 (df)^2\, dt \;,$$

where we define

$$B = \frac{1}{\lambda}\, d\nu\, du\, \alpha\, e\, \psi(u) \sin u \,.$$

According to the energy principle we must have

$$E_P = E_L \;,$$

and it follows that

$$B'\, df' = B\, df$$

or

$$\frac{1}{\lambda'}\, d\nu\, du\, \alpha' e' \left(\frac{\lambda'}{\lambda}\right)^2 \frac{1}{\beta^2} \frac{\cos u}{\cos u'} \psi(u') \sin u \cdot \beta^2 = \frac{1}{\lambda}\, d\nu\, du\, \alpha\, e\, \psi(u) \sin u \,;$$

or if one inserts here the previously obtained value of $\alpha' e'/\alpha e$,

$$\frac{\psi(u')}{\psi(u)} = \sqrt{\frac{\cos u'}{\cos u}}$$

or

$$\frac{\psi(u')}{\sqrt{\cos u'}} = \frac{\psi(u)}{\sqrt{\cos u}} \,.$$

Indeed, u and u' are dependent on each other in this special case; however, one can assign, by varying β (changing the system), every arbitrary value of u to the same u', so it is valid that

$$\frac{\psi(u')}{\sqrt{\cos u'}} = \frac{\psi(u_1)}{\sqrt{\cos u_1}} = \frac{\psi(u_2)}{\sqrt{\cos u_2}} \;;$$

therefore, we must have

$$\psi(u') = \sqrt{\cos u'}\,. \tag{22}$$

§15. Expression of light disturbance at the observation point
If the radiating surface element radiates according to Lambert's law,

$$\alpha = \frac{\text{const}}{e}\sqrt{\cos u}\,,$$

considering the derived relationships (Eqs. 21 and 22)

$$\sigma(u) = \frac{\text{const}}{e}\sqrt{\frac{\cos u}{\cos u'}}$$
$$\psi(u') = \sqrt{\cos u'}\,,$$

Eq. 18 for the light disturbance at P_1 finally takes the form

$$s = \frac{k}{\lambda}\int\limits_{I}\frac{\cos u}{e^2}\,d\varphi\,\sin 2\pi\left(\frac{t}{T} - \frac{x\xi + y\eta}{e\lambda}\right)$$

or, since $d\varphi\cos u = d\xi\,d\eta$,

$$s = \frac{k}{\lambda}\int\limits_{I}\frac{d\xi\,d\eta}{e^2}\sin 2\pi\left(\frac{t}{T} - \frac{x\xi + y\eta}{e\lambda}\right)\,, \tag{23}$$

in which the integration extends over the projection of the boundary surface on the $\xi\eta$-plane.

x and y are the coordinates of L_1, the point, with respect to the system, conjugate to the observation point P_1. The intensity at P_1 is given by

$$J_{P_1} = \overline{s^2}\,df\;. \tag{23a}$$

One can of course, in the calculation of the light disturbance at P_1, also use integral s', which extends over surface II behind the system. Then,

$$s' = \frac{k'}{\lambda'}\int\limits_{II}\frac{d\xi'\,d\eta'}{e'^2}\sin 2\pi\left(\frac{t}{T} - \frac{x'\xi' + y'\eta'}{e'\lambda'}\right)\;; \tag{24}$$

x' and y' are the coordinates of the observation point P_1. The intensity at P_1 is then

$$J_{P_1} = \overline{s'^2}\, df' = \beta^2 \overline{s'^2}\, df \ . \tag{24a}$$

Whereas one reaches the final expression of s or s' via a somewhat laborious determination of factors σ and ψ, which of course allows a deeper insight into the energy relationships, one obtains an expression in a shorter way by means of the Kirchhoff principle, which, for u' not too large, agrees with s' found above.

§16. Determination of light disturbance at the observation point using the Kirchhoff principle

Again let the intensity at $d\varphi$ of the radiation originating from element df (Fig. 23) be

$$J_{d\varphi} = \text{const}\frac{\cos u \cdot df}{e^2} \ .$$

According to the electromagnetic theory of light, up to a constant, this intensity must be identical with the time average of the governing electric field at the location of $d\varphi$; that is,

$$J_{d\varphi} = \overline{\mathfrak{E}^2} = \text{const}\frac{\cos u \cdot df}{e^2} \ . \tag{25}$$

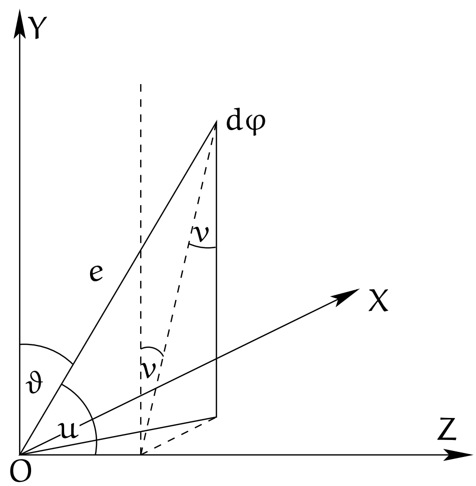
One can replace this unpolarized radiation of the surface element df with the radiation of a dipole whose axis stands perpendicularly to the axis of the system and rotates in the plane of element df about the system axis.

Proof: it is generally known that the electric field at $d\varphi$ generated by a *stationary* dipole at df is[xxxv]

$$\mathfrak{e} = \frac{A}{e}\sin\vartheta\cos 2\pi\left(\frac{t}{T} - \frac{e}{\lambda}\right) \ ,$$

provided that e is large compared to λ. ϑ is the angle that the radius vector e (Fig. 24) forms with the axis OY of the dipole at O. If one

Figure 24



introduces polar coordinates e, u, v around the system axis OZ, then

$$\cos\vartheta = \sin u \cdot \cos v$$

or

$$\sin\vartheta = \sqrt{1-\sin^2 u \cos^2 v}\,,$$

in which v, as the dipole *rotates*, varies between 0 and 2π. The average value of the electric field is therefore[xxxvi]

$$\mathfrak{E} = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{e}\,dv = \frac{A}{e}\cos 2\pi\left(\frac{t}{T}-\frac{e}{\lambda}\right)\cdot\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\sin^2 u\cos^2 v}\,dv$$

$$= \frac{A}{e}\cos 2\pi\left(\frac{t}{T}-\frac{e}{\lambda}\right)\cos^2\frac{u}{2}$$

$$\times\left\{1+\left(\frac{1}{2}\right)^2\tan^4\frac{u}{2}+\left(\frac{1}{2\cdot 4}\right)^2\tan^8\frac{u}{2}+\cdots\right\}.$$ [xxxvii]

If $\mathfrak{u}$ is not too large, we can restrict ourselves to the first term in the series, because even for $\mathfrak{u} = 20^\circ$, the value of the second term is only 0.00024. We therefore obtain

$$\begin{aligned}\mathfrak{E} &= \frac{A}{e}\cos 2\pi\left(\frac{t}{T}-\frac{e}{\lambda}\right)\cos^2\frac{\mathfrak{u}}{2}\\ &= \frac{A}{e}\cos 2\pi\left(\frac{t}{T}-\frac{e}{\lambda}\right)\cdot\frac{1+\cos\mathfrak{u}}{2}\,.\end{aligned}$$

For not-too-large $\mathfrak{u}$ we can replace the factor $\frac{1+\cos\mathfrak{u}}{2}$ by $\sqrt{\cos\mathfrak{u}}$; *even with* $\mathfrak{u} = 20^\circ$ *these two values agree to the third decimal place.*[xxxviii] Therefore, we finally obtain

$$\begin{aligned}\mathfrak{E} &= \frac{A}{e}\cos 2\pi\left(\frac{t}{T}-\frac{e}{\lambda}\right)\cdot\sqrt{\cos\mathfrak{u}}\,,\\ \mathfrak{E}^2 &= \frac{1}{2}\frac{A^2}{e^2}\cos\mathfrak{u}\,.\end{aligned}$$

Therefore, if we set according to Eq. 25

$$A^2 = 2\cdot\mathrm{const}\cdot df\ ,$$

we have proved that *one can replace the radiating surface element* df *according to the cosine law with the radiation of a rotating dipole.*

If the convergence angle $\mathfrak{u}'$ in the image space is not too large, as we assume, then we are justified to set at the location of $d\varphi'$,

$$\mathfrak{e}' = \frac{A'}{e'}\sin\vartheta'\cos 2\pi\left(\frac{t}{T}+\frac{e'}{\lambda'}\right)$$

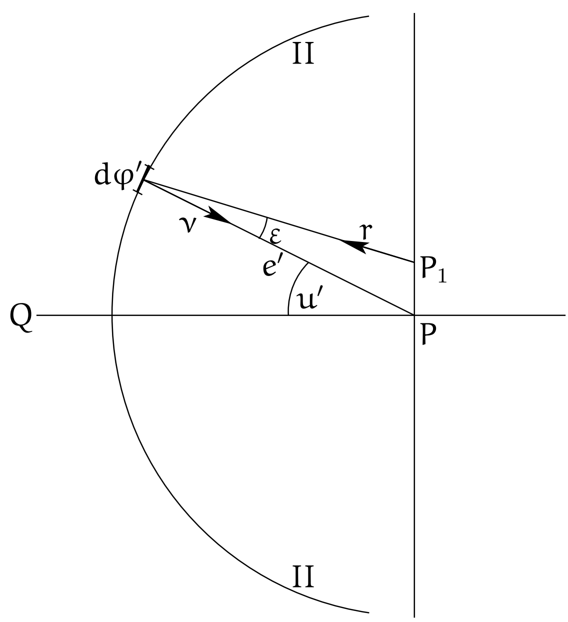
or

$$\mathfrak{e}' = \frac{A'}{e'}\sqrt{1-\sin^2\mathfrak{u}'\cos^2\nu}\cos 2\pi\left(\frac{t}{T}+\frac{e'}{\lambda'}\right)\ ,$$

where ϑ' for the image space has the analogous meaning as ϑ for the object space, and denotes the angle between e' and the axis of the dipole perpendicular to QP. To obtain *unpolarized* surface radiation, we must subsequently still form the average value of this expression over all ν, from 0 to 2π.

To apply the Kirchhoff principle to a vector, we must insert, as surface values, the values of those *vector components* and their derivatives with respect to the normal of the integration surface that are *parallel to the resulting vector at the observation point*. If we assume the bounding aperture to be *symmetrical* with respect to axis QP, the resulting vector $\mathfrak{e}'$ of the field at P (Fig. 25) and at paraxial point P_1, generated by the

Figure 25



stationary dipole, has necessarily the direction parallel to the dipole axis and perpendicular to axis QP. At $d\varphi'$, however, $\mathfrak{e}'$ is tangential to spherical surface II and therefore forms the angle $\frac{\pi}{2} - \vartheta'$ with the direction of the resulting vector at P_1.

Thus, as surface values, we take

$$\mathfrak{e}' \cos\left(\frac{\pi}{2} - \vartheta'\right) = \mathfrak{e}' \sin\vartheta' = \mathfrak{e}'\sqrt{1 - \sin^2 u' \cos^2 \nu}$$

and their derivatives with respect to ν.

If the dipole *rotates*, we form the average value of these magnitudes with respect to ν and obtain

$$\begin{aligned}
\mathfrak{E}' &= \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{e}'\sqrt{1 - \sin^2 u' \cos^2 \nu}\, d\nu \\
&= \frac{1}{2\pi}\frac{A'}{e'} \cos 2\pi\left(\frac{t}{T} + \frac{e'}{\lambda'}\right)\int_0^{2\pi}(1 - \sin^2 u' \cos^2 \nu)\, d\nu \\
&= \frac{A'}{e'} \cos 2\pi\left(\frac{t}{T} + \frac{e'}{\lambda'}\right)\left(1 - \frac{1}{2}\sin^2 u'\right)\ ;
\end{aligned}$$

and since u' is assumed to be small,

$$\mathfrak{E}' = \frac{A'}{e'} \cos 2\pi\left(\frac{t}{T} + \frac{e'}{\lambda'}\right) \cdot \cos u'\ . \tag{26}$$

To apply Kirchhoff's law on $\mathfrak{E}'$, we still have to show that $\mathfrak{E}'$ is a solution of the wave equation (Eq. 12), which takes on, with the introduction of polar coordinates and especially for the present case, the following form:[xxxix]

$$\frac{1}{a'^2}\frac{\partial^2 \mathfrak{E}'}{\partial t^2} = \frac{1}{e'}\frac{\partial^2 (e'\mathfrak{E}')}{\partial e'^2} + \frac{1}{e'^2 \sin u'}\frac{\partial\left(\sin u' \frac{\partial \mathfrak{E}'}{\partial u'}\right)}{\partial u'}\ .$$

Here, a' is the velocity of propagation of the waves in the image space.

A solution of this equation is[xl]

$$\mathfrak{E}' = \frac{\text{const}}{e'} \cos u' \left\{\cos 2\pi\left(\frac{t}{T} + \frac{e'}{\lambda'}\right) - \frac{\lambda'}{2\pi e'} \sin 2\pi\left(\frac{t}{T} + \frac{e'}{\lambda'}\right)\right\}\ ,$$

which, since e' is large compared to λ', reduces to the expression identical to Eq. 26,

$$\mathfrak{E}' = \frac{\text{const}}{e'} \cos u' \cdot \cos 2\pi \left(\frac{t}{T} + \frac{e'}{\lambda'} \right) .$$

With this it has been shown that $\mathfrak{E}'$ is a solution of the wave equation for the case treated here and therefore can be inserted in place of φ in Eq. 13 of the Kirchhoff principle.

If one introduces once again s' via Eq. 24a,

$$J_{P_1} = \overline{\mathfrak{E}'^2} = \overline{s'^2} \cdot \overline{df'} \,,$$

after easy calculation,[xli] if one replaces r with e' in the amplitude and $\frac{1+\cos u}{2}$ with 1, one obtains

$$\begin{aligned} s' &= \frac{k'}{\lambda'} \int\limits_{II} \frac{d\varphi' \cos u'}{e'^2} \sin 2\pi \left(\frac{t}{T} + \frac{x'\xi' + y'\eta'}{e'\lambda'} \right) \\ &= \frac{k'}{\lambda'} \int\limits_{II} \frac{d\xi'\, d\eta'}{e'^2} \sin 2\pi \left(\frac{t}{T} + \frac{x'\xi' + y'\eta'}{e'\lambda'} \right) , \end{aligned}$$

which is exactly the above derived expression (Eq. 24).

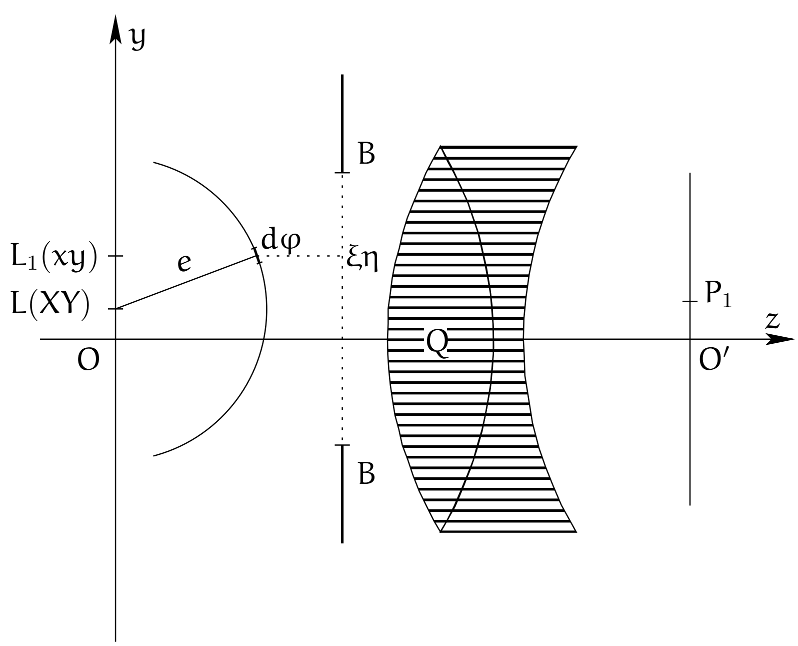
It should be pointed out once more that one obtains the "effective piece of boundary surface I" as one draws from the luminous point or surface element all possible rays toward the boundary points on the entrance pupil. The entirety of the intersections of these rays with the spherical surface I form the boundary of the "effective piece." Integration in the expression of s is extended over the projection of this "effective piece" onto the $\xi\eta$-plane.

§17. Calculation of diffraction on an aperture of specific form for points in the plane conjugate to the object plane in the presence of a luminous surface element

We choose the form of the diffracting aperture in such a way that *the projection of the effective piece of the boundary surface onto the $\xi\eta$-plane is* *a rectangle*. The diffracting aperture in this case, as easily calculated, is bordered by four hyperbolae and approximates better the form of a rectangle the smaller the dimensions of the aperture.

Let OO' (Fig. 26) be the optical axis of the imaging system Q, and O be the origin of the rectangular coordinate system whose z-axis coincides with the optical axis; let the y-axis be pointed toward the top, and the x-axis toward the back. Let the xy-plane be the object plane containing a luminous surface element df at L with coordinates XY. Let the plane perpendicular to OO' and containing O' be the image plane conjugate to the object plane, and the observation point lie at P_1. Let the ray-limiting aperture be represented by the physical

Figure 26



and perpendicular-to-the-z-axis standing diaphragm BB in front of the imaging system Q. Let the radius of the L-centered sphere that we choose as the boundary surface be e; let the luminous element be always so close to the axis that the quadratic terms in x and X, and y and Y can be neglected. Let $d\varphi$ be an element of the boundary surface and its projection on the plane of the diaphragm have the coordinates $\xi\eta$. Then the light disturbance at point P_1 situated close to the z-axis is given by the expression

$$s = \frac{k}{\lambda} \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} \frac{d\xi \, d\eta}{e^2} \sin 2\pi \left(\frac{t}{T} - \frac{x'\xi + y'\eta}{e\lambda} \right) , \tag{27}$$

where

$$\begin{aligned} x' &= x - X \\ y' &= y - Y \end{aligned} \tag{27a}$$

are the coordinates of point L_1, which is conjugate to the observation point P_1, if one refers to them[15] using the luminous element at L as the starting point, and the integration is extended over the rectangular projection of the effective pieces of the boundary surface. One sets

$$\begin{aligned} \xi' &= \xi/e \\ \eta' &= \eta/e \end{aligned} \tag{28}$$

and Eq. 27 becomes

$$s = \frac{k}{\lambda} \int_{\xi'_1}^{\xi'_2} \int_{\eta'_1}^{\eta'_2} d\xi' \, d\eta' \sin 2\pi \left(\frac{t}{T} - \frac{x'\xi' + y'\eta'}{\lambda} \right) . \tag{29}$$

[15]It should be emphasized that these *relative* coordinates $x'y'$ are not identical with the absolute coordinates $x'y'$ of P_1 used in previous paragraphs.

If one decomposes the sine function into its components,

$$\sin 2\pi\left(\frac{t}{T}-\frac{x'\xi'}{\lambda}\right)\cos 2\pi\frac{y'\eta'}{\lambda}-\cos 2\pi\left(\frac{t}{T}-\frac{x'\xi'}{\lambda}\right)\sin 2\pi\frac{y'\eta'}{\lambda}\ ,$$

one can carry out the integrations with respect to ξ' and η' separately and obtain[xlii]

$$s=\frac{k}{\lambda}\frac{\sin 2\pi x'\frac{\xi_2'-\xi_1'}{2\lambda}}{\pi\frac{x'}{\lambda}}\cdot\frac{\sin 2\pi y'\frac{\eta_2'-\eta_1'}{2\lambda}}{\pi\frac{y'}{\lambda}}\cdot\sin 2\pi\left(\frac{t}{T}-\frac{x'(\xi_2'+\xi_1')+y'(\eta_2'+\eta_1')}{2\lambda}\right)\ .$$

The two integrations will no longer be independent of each other if the projection of the effective boundary surface deviates from the shape of the rectangle.

A simplification occurs if the aperture lies symmetrically with respect to the *z*-axis. In this case,

$$\frac{\xi_1'+\xi_2'}{2}=0 \text{ and } \frac{\eta_1'+\eta_2'}{2}=0\ .$$

If one further sets

$$\xi_2-\xi_1=2\alpha \text{ and } \eta_2-\eta_1=2\beta\ ,$$

where α and β denote the half width and height of the projection of the boundary surface, we have

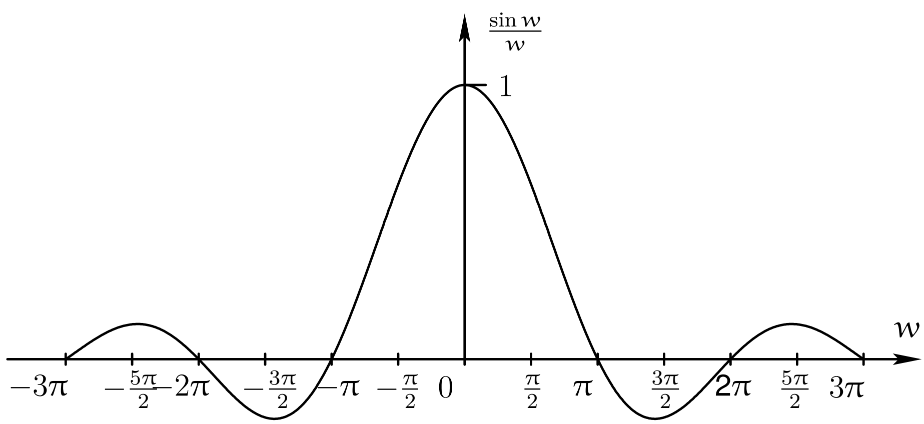
$$\frac{\xi_2'-\xi_1'}{2}=\frac{\alpha}{e}=\alpha' \text{ and } \frac{\eta_2'-\eta_1'}{2}=\frac{\beta}{e}=\beta'\ ,$$

where α' and β' are the sines of the aperture angle of the half width and height of the projection of the symmetrical diaphragm. We then have

$$s=\frac{k}{\lambda}4\alpha'\beta'\frac{\sin 2\pi\frac{x'\alpha'}{\lambda}}{2\pi\frac{x'\alpha'}{\lambda}}\cdot\frac{\sin 2\pi\frac{y'\beta'}{\lambda}}{2\pi\frac{y'\beta'}{\lambda}}\sin 2\pi\frac{t}{T}\ . \tag{30}$$

The amplitude of the oscillation s, whose phase is given by $\sin 2\pi\frac{t}{T}$, consists of, apart from a constant, the product of two factors of the form $F(w) = \frac{\sin w}{w}$. The graph of this function of w is indicated in Fig. 27. For $w = \pm\mathfrak{a}\pi$ ($\mathfrak{a} = 1, 2, 3\ldots$), $F(w) = 0$; for $w = 0$, $F(w)$ takes on the undetermined expression $0/0$, whose true value is one.

Figure 27



Without further ado, one can see from the form of the function that the amplitude has its maximum at $w = 0$ and decreases gradually from there toward both sides symmetrically with increasing $|w|$. Whereas the first factor

$$\frac{\sin 2\pi\frac{x'\alpha'}{\lambda}}{2\pi\frac{x'\alpha'}{\lambda}}$$

depicts the amplitude in directions parallel to the x-axis, the second factor,

$$\frac{\sin 2\pi\frac{y'\beta'}{\lambda}}{2\pi\frac{y'\beta'}{\lambda}},$$

independent from the first, reproduces the course of the amplitude in directions parallel to the y-axis. One thus sees that the amplitude of the oscillation is arranged in a checkered way and symmetrically with respect to the lines $x' = 0$ and $y' = 0$ (or $x = X$ and $y = Y$). The amplitude is zero (minimum) on lines

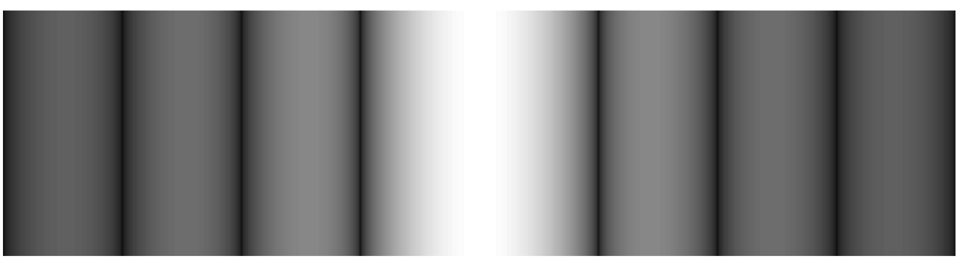
$$x' = \pm a \frac{\lambda}{2\alpha'} \; (a = 1, 2, 3 \ldots)$$

and

$$y' = \pm a \frac{\lambda}{2\beta'} \; (a = 1, 2, 3 \ldots) \, .$$

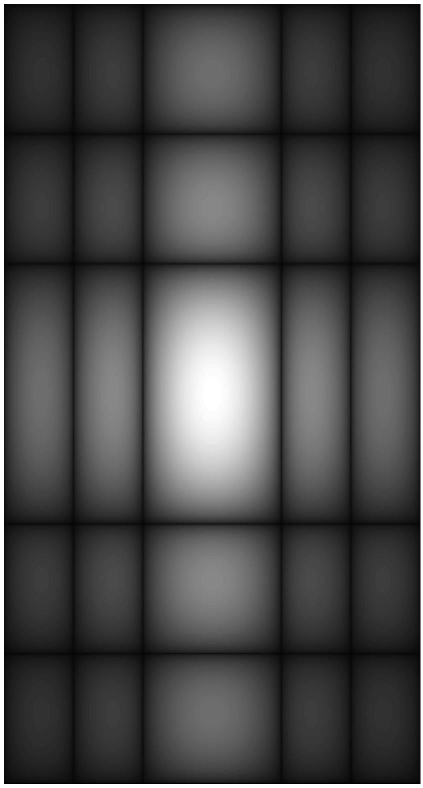
These lines form a system of rectangles in which the amplitude increases gradually from the sides to the middle and has its maximum there (the cross point of the diagonals). The closer the rectangle is situated to the center of the pattern, the greater the maximum. In the central rectangle, the amplitude reaches its absolute maximum (Fig. 28) at the position of the luminous element ($x' = 0$, $y' = 0$).

Figure 29



One can see from the equations for the lines of minima that the smaller the dimension of α', defined for the angular "width" of the diffracting aperture, the farther the lines parallel to the y-axis move

Figure 28



away from each other, and the distance of the lines parallel to the x-axis depends on β' (angular "height") in the same way.

If, for example, the width (α) is negligible compared to the height (β), i.e., the diffracting aperture is formed by a vertical narrow slit, the distribution of the amplitude then takes on the appearance sketched in Fig. 29.[xliii] The intensity distribution of the actually observed diffraction phenomenon emerges from the obtained amplitude distribution if one squares the amplitude at every location, for in general, $J = \overline{s^2}\,df$.